Generalized fractional smoothness and Lp-variation of BSDEs with non-Lipschitz terminal condition

Christel Geiss, Stefan Geiss, Emmanuel Gobet

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Christel Geiss, Stefan Geiss* and Emmanuel Gobet †

Department of Mathematics
University of Innsbruck
Technikerstraße 13/7
A-6020 Innsbruck
Austria
christel.geiss@uibk.ac.at and stefan.geiss@uibk.ac.at

Centre de Mathématiques Appliquées‡
Ecole Polytechnique
F-91128 Palaiseau Cedex
France
emmanuel.gobet@polytechnique.edu

March 1, 2011

Abstract

We relate the L_p-variation, $2 \leq p < \infty$, of a solution of a backward stochastic differential equation with a path-dependent terminal condition to a generalized notion of fractional smoothness. This concept of fractional smoothness takes into account the quantitative propagation of singularities in time.

MSC2010: 60H10, 46B70

*The author is supported by the Project 133914 of the *Academy of Finland*.

†This research is part of the Chair *Financial Risks* of the *Risk Foundation*, the Chair *Derivatives of the Future* and the Chair *Finance and Sustainable Development*.

‡A significant part of this work was done when the third author was affiliated to Grenoble Institute of Technology.

Introduction

During the last years the concept of fractional smoothness in the sense of function spaces has been used in the theory of stochastic processes to analyze approximation and variational properties. It turned out that phenomena known for special examples can be explained in terms of fractional smoothness. For example, approximation properties of certain stochastic integrals can be explained by the fractional smoothness of the integral itself, see [10, 11]. Similarly, variational properties of backward stochastic differential equations (BSDEs) can be upper bounded in case that the fractional smoothness of the terminal condition is known. To explain the latter aspect consider the BSDE

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

with a Lipschitz generator f, where $X = (X_t)_{t\in[0,T]}$ is a forward diffusion, and define the L_p-variation

$$\mathrm{var}_p(\xi, f, \tau) := \sup_{i=1,\ldots,n} \sup_{t_{i-1}<s\le t_i} \|Y_s - Y_{t_{i-1}}\|_p + \left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|Z_t - \overline{Z}_{t_{i-1}}\|_p^2 dt \right)^{\frac{1}{2}}$$

where $\tau = (t_i)_{i=0}^n$ is a deterministic time-net $0 = t_0 < \cdots < t_n = T$,

$$\overline{Z}_{t_{i-1}} := \frac{1}{t_i - t_{i-1}} \mathbb{E}\left[\int_{t_{i-1}}^{t_i} Z_s ds | \mathcal{F}_{t_{i-1}} \right],$$

and where $2 \le p < \infty$, which we will assume throughout this paper. Note that by interchanging the L_p- and L_2-norms (where we use $p \ge 2$) and using the Burkholder-Davis-Gundy inequality, the L_p-distance between the stochastic integral $\int_0^T Z_s dW_s$ and its discrete counterpart $\sum_{i=1}^n \overline{Z}_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})$ is upper bounded by a multiple of $\left(\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|Z_t - \overline{Z}_{t_{i-1}}\|_p^2 dt \right)^{\frac{1}{2}}$. Hence the quantity $\mathrm{var}_p(\xi, f, \tau)$ is stronger compared to what is needed to quantify the discretization of the stochastic integral term of our BSDE. Besides the fact that this variation gives a strong insight into the quantitative behavior of the BSDE, in particular $\mathrm{var}_2(\xi, f, \tau)$ was used to describe the error in adapted backward Euler schemes for $\xi = g(X_T)$ with g being a Lipschitz function; see [5, 23] for implicit schemes and [15, 16] for explicit schemes possibly with jump processes. In [14, Theorems 3.1 and 3.2] upper bounds for

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|Z_t - \overline{Z}_{t_{i-1}}\|_2^2 dt$$

were obtained for $\xi = g(X_T)$ satisfying

$$\mathbb{E}|g(X_T) - \mathbb{E}(g(X_T)|\mathcal{F}_t)|^2 \le c^2 (T-t)^\theta$$

for some $0 < \theta \leq 1$, where g is not assumed to be a Lipschitz function. On the other hand, path-dependent settings without taking into account fractional smoothness were considered, for example, in [19, 20, 25] In this paper, results are generalized and extended into the following directions:

- We consider a path-dependent setting by terminal conditions of the form

$$\xi = g(X_{r_1}, ..., X_{r_L})$$

  with $0 = r_0 < \cdots < r_L = T$, where g is not necessarily a Lipschitz function and introduce a corresponding path-dependent fractional smoothness in the Malliavin sense. This concept of smoothness extends the classical concepts, based on real interpolation, to a time-dependent one taking care about the propagation of smoothness in time. In the classical case one would assign to a random variable ξ some $0 < \theta \leq 1$ which describes the fractional smoothness of ξ while here we assign to the parameters (ξ, f) of our BSDE a *vector* $\Theta = (\theta_1, ..., \theta_L)$, where θ_l stands for the local smoothness of the BSDE at time r_l. It turns out that this vector is completely characterized by the L_p-variation of Y and Z. In case our terminal condition depends on X_T only our generalized smoothness coincides with earlier approaches from, for example, [10] and [14].

- Instead of the L_2-variation we consider the stronger L_p-variation with $2 \leq p < \infty$. In addition, the integrated Z-variation $\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|Z_t - \overline{Z}_{t_{i-1}}\|_2^2 dt$ is replaced by the variation $\|Z_s - Z_t\|_p$ with s and t being fixed, and the L_p-variation of the process Y is included as well. To our knowledge the weaker criterion for $0 < p < 2$ in the context of this paper has not been considered yet and might require different arguments as some of our proofs rely on the condition that $p \geq 2$.

- We provide equivalences showing that the results are sharp.

- In Corollary 2.4 we show, given the terminal condition $\xi = g(X_{r_1}, ..., X_{r_L})$ has a certain fractional smoothness, how to obtain time-nets τ^n of cardinality $Ln + 1$ such that

$$\sup_n \sqrt{n}\, \mathrm{var}_p(\xi, f, \tau^n) < \infty.$$

  These time-nets compensate the possible singularities of the Z-process when approaching a time-point r_l from the left.

Organization of the paper. After introducing the setting in Section 1, we formulate in Section 1.2 our concept of functional fractional smoothness of a BSDE and characterize this smoothness in various ways. Here we partly transfer the results from [10] and [14] from the case $\xi = g(X_T)$ to the path-dependent one. In Section 2.2 we present two sufficient conditions for our fractional smoothness. The point of these two conditions (Corollary 2.6 and Theorem 2.10) is that they only involve the terminal condition ξ and do not use the solution Y nor the generator f of our BSDE. The proofs of the main results are contained in Section 3.

Some notation. Given a vector $x \in \mathbb{R}^d$ we denote by $|x|$ its Euclidean norm, for a linear operator $D \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ the symbol $|D|$ stands for the Hilbert-Schmidt norm, where $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the standard Euclidean structure. Given $D(t,x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with $(t,x) \in [0,T] \times \mathbb{R}^d$ and $0 < T < \infty$, we use

$$\|D\|_\infty := \sup_{x \in \mathbb{R}^d, t \in [0,T]} |D(t,x)|.$$

Finally, $B(\eta_1, \eta_2) := \int_0^1 x^{\eta_1 - 1}(1-x)^{\eta_2 - 1} dx$ where $\eta_1, \eta_2 > 0$, will denote the Beta-function.

1 Setting and basic concepts

1.1 Forward-backward stochastic differential equations

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $T > 0$, $d \geq 1$ and a d-dimensional standard Brownian motion $W = (W_t)_{t \in [0,T]}$ with $W_0 \equiv 0$. Furthermore, we assume that $(\mathcal{F}_t)_{t \in [0,T]}$ is the augmentation of the natural filtration of W.

The forward equation. Let

$$X_t = x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s$$

with $x_0 \in \mathbb{R}^d$, where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ satisfy the following conditions:

$(A_{b,\sigma})$ We have $b, \sigma \in C_b^{0,2}([0,T] \times \mathbb{R}^d)$, where the derivatives up to order two are taken with respect to the space-variables and, for some $\gamma \in (0,1]$, are assumed to be γ-Hölder continuous (w.r.t. the parabolic metric) on all compact subsets of $[0,T] \times \mathbb{R}^d$. Moreover, there is a $\delta > 0$ such that $\langle Ax, x \rangle \geq \delta |x|^2$ for $x \in \mathbb{R}^d$ and b and σ are $\frac{1}{2}$-Hölder continuous in time, uniformly in space.

We work with the usual stochastic flow $(X_s^{t,x})_{s,t \in [0,T], x \in \mathbb{R}^d}$ that solves for $(t,x) \in [0,T] \times \mathbb{R}^d$ the SDE $X_s = x$ on $[0,t]$ and $dX_s^{t,x} = \sigma(s, X_s^{t,x}) dW_s^t + b(s, X_s^{t,x}) ds$ on $[t,T]$, where $W_s^t := W_s - W_t$ and the augmented natural filtration $(\mathcal{F}_s^t)_{s \in [t,T]}$ of $(W_s^t)_{s \in [t,T]}$ is used (i.p. $X = X^{0,x_0}$). With our assumptions we can assume that $(X_s^{t,x})_{s,t \in [0,T], x \in \mathbb{R}^d}$ is a continuous process in (s,t,x).

If $g : \mathbb{R}^d \to \mathbb{R}$ is a polynomially bounded Borel function, $0 < R \leq T$, and

$$F(t,x) := \mathbb{E} g(X_R^{t,x}) \quad \text{for} \quad 0 \leq t \leq R, \tag{1}$$

then $F \in C^{1,2}([0,R) \times \mathbb{R}^d)$ and

$$\frac{\partial}{\partial t}F(t,x) + \frac{1}{2}\left\langle A(t,x), D^2F(t,x)\right\rangle + \langle b(t,x), \nabla_x F(t,x)\rangle = 0$$

by Proposition B.1 below where

$$D^2 := \left(\frac{\partial^2}{\partial x_i \partial x_j}\right)_{i,j=1}^d .$$

The standard tail estimates for the transition density Γ are re-called in Proposition B.1. They ensure that $\frac{\partial}{\partial t}\nabla_x F$, $\nabla_x \frac{\partial}{\partial t}F$ and $D_x^m F$ with $|m| \le 3$ exist and are continuous on $[0,R) \times \mathbb{R}^d$. For $0 \le t \le r < R \le T$ one has that, a.s.,

$$\begin{aligned}
\nabla_x F(r, X_r^{t,x}) &= \mathbb{E}\left(g(X_R^{t,x}) N_R^{r,1,(t,x)} | \mathcal{F}_r^t\right), \\
D^2 F(r, X_r^{t,x}) &= \mathbb{E}\left(g(X_R^{t,x}) N_R^{r,2,(t,x)} | \mathcal{F}_r^t\right)
\end{aligned}$$

for the Malliavin weights $N_R^{r,i,(t,x)}$ that satisfy, for any given $0 < q < \infty$, that

$$\left[\mathbb{E}\left(\left|N_R^{r,i,(t,x)}\right|^q |\mathcal{F}_r^t\right)\right]^{\frac{1}{q}} \le \frac{\kappa_q}{(R-r)^{\frac{i}{2}}} \; a.s. \; \text{ and } \; \mathbb{E}\left(N_R^{r,i,(t,x)} | \mathcal{F}_r^t\right) = 0 \; a.s.$$

for $i = 1,2$ and all $0 \le t \le r < R \le T$ with a constant $\kappa_q > 0$ independent from (t,r,R,x) (see [17], [14, Proof of Lemma 1.1] and Remark B.2 below). A typical application of these estimates are the crucial inequalities

$$\|\nabla_x F(r, X_r^{t,x})\|_p \le \kappa_{p'} \frac{\|g(X_R^{t,x}) - \mathbb{E}(g(X_R^{t,x})|\mathcal{F}_r^t)\|_p}{\sqrt{R-r}}, \tag{2}$$

$$\|D^2 F(r, X_r^{t,x})\|_p \le \kappa_{p'} \frac{\|g(X_R^{t,x}) - \mathbb{E}(g(X_R^{t,x})|\mathcal{F}_r^t)\|_p}{R-r}, \tag{3}$$

for $1 < p, p' < \infty$ with $1 = (1/p) + (1/p')$.

The backward equation. We are interested in the backward equation

$$Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \quad \text{ for } t \in [0,T] \quad a.s.$$

and assume the following conditions:

(A_f) The function $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuous in (t,x,y,z) and continuously differentiable in x, y and z with uniformly bounded derivatives. In particular, there are $K_f > 0$ and $L_f > 0$ such that

$$\begin{aligned}
|f(s,x_1,y_1,z_1) - f(s,x_2,y_2,z_2)| &\le L_f[|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|], \\
|f(s,x,y,z)| &\le K_f + L_f(|x| + |y| + |z|).
\end{aligned}$$

(A_g) There are $\mathcal{R} = \{r_0, ..., r_L\}$ with $0 = r_0 < r_1 < \cdots < r_L = T$ and a measurable function of at most polynomial growth $g : (\mathbb{R}^d)^L \to \mathbb{R}$ such that

$$\xi := g(X_{r_1}, ..., X_{r_L}).$$

In this setting, the solution (Y, Z) to the above BSDE is uniquely defined in any L_p-space for $1 < p < \infty$; see [6, Theorem 4.2]. Additionally, we assume in the paper that the solution (Y, Z) is realized such that, on $[r_{l-1}, r_l)$,

$$Y_t = u_l(\overline{X}_{l-1}; t, X_t) \quad \text{and} \quad Z_t = v_l(\overline{X}_{l-1}; t, X_t)\sigma(t, X_t),$$

where we set $\overline{X}_{l-1} := (X_{r_1}, \ldots, X_{r_{l-1}})$. The above functions u_l and v_l are well defined due to the next proposition, which is an extension of [24, Theorem 3.2] and follows from Lemma A.2, see also [19].

Proposition 1.1. *Assume that $(A_{b,\sigma}), (A_f)$ and (A_g) are satisfied. Then, for $l = 1, ..., L$ there exist measurable $u_l : (\mathbb{R}^d)^{l-1} \times [r_{l-1}, r_l) \times \mathbb{R}^d \to \mathbb{R}$ and $v_l : (\mathbb{R}^d)^{l-1} \times [r_{l-1}, r_l) \times \mathbb{R}^d \to \mathbb{R}^{1\times d}$ and Borel sets $D_l \subseteq \mathbb{R}^{d(l-1)}$, $l = 2, ..., L$, such that D_l^c is of Lebesgue measure zero, and such that*

(i) *$u_l(\overline{x}_{l-1}; \cdot, \cdot) : [r_{l-1}, r_l) \times \mathbb{R}^d \to \mathbb{R}$ is continuous and continuously differentiable w.r.t. the space variable with $\nabla_x u_l(\overline{x}_{l-1}; t, x) = v_l(\overline{x}_{l-1}; t, x)$, where $\overline{x}_{l-1} = (x_1, \ldots, x_{l-1})$,*

(ii) *there are $\alpha_l, q_{l,1}, ..., q_{l,l} \in [1, \infty)$ such that*

$$\sup_{t\in[r_{l-1},r_l)} |u_l(\overline{x}_{l-1}; t, x)| + \sup_{t\in[r_{l-1},r_l)} \sqrt{r_l - t}|v_l(\overline{x}_{l-1}; t, x)| \\ \leq \alpha_l(1 + |x_1|^{q_{l,1}} + \cdots + |x_{l-1}|^{q_{l,l-1}} + |x|^{q_{l,l}}),$$

(iii) *for all $l = 1, ..., L$, $x_1, ..., x_{l-1}, x \in \mathbb{R}^d$ and $r_{l-1} \leq s < r_l$ the triplet*

$$\left(X_t^{s,x}, u_l(\overline{x}_{l-1}; t, X_t^{s,x}), v_l(\overline{x}_{l-1}; t, X_t^{s,x})\sigma(t, X_t^{s,x})\right)_{t\in[s,r_l)}$$

solves the BSDE with generator f and terminal condition

$$u_l(\overline{x}_{l-1}; r_l, X_{r_l}^{s,x})$$

where

$$u_l(\overline{x}_{l-1}; r_l, x) := \begin{cases} u_{l+1}(\overline{x}_{l-1}, x; r_l, x)\chi_{D_l}(\overline{x}_{l-1}) & : \quad 2 \leq l < L, \\ g(\overline{x}_{l-1}, x)\chi_{D_l}(\overline{x}_{l-1}) & : \quad l = L. \end{cases}$$

and $u_1(r_1, x) := u_2(x; r_1, x)$.

In the above proposition we used the convention that $h(\overline{x}_0; \cdot) := h(\cdot)$. It should be noted that by Proposition 1.1 we modify at each level $l = 2, ..., L$ the functional for the Y-process on a nullset. However, because of

$$\mathbb{P}(X_{r_1} \in D_2, ..., (X_{r_1}, ..., X_{r_{L-1}}) \in D_L) = 1, \tag{4}$$

this does not affect the L_p-solution of our BSDE so that Proposition 1.1 is sufficient for our purpose.

Piece-wise linearization of the backward equation. We let $F_l(\overline{x}_{l-1};\cdot,\cdot):[r_{l-1},r_l]\times\mathbb{R}^d\to\mathbb{R}$ be given by

$$F_l(x_1,...,x_{l-1};t,x)=F_l(\overline{x}_{l-1};t,x):=\mathbb{E}u_l(x_1,...,x_{r_{l-1}};r_l,X_{r_l}^{t,x}).$$

The function F_l solves the backward PDE

$$\frac{\partial F_l}{\partial t}(\overline{x}_{l-1};t,x)+\frac{1}{2}\left\langle A(t,x),D^2F_l(\overline{x}_{l-1};t,x)\right\rangle+\langle b,\nabla_x F_l(\overline{x}_{l-1};t,x)\rangle=0$$

on the interval $[r_{l-1},r_l)$ for fixed $x_1,...,x_{l-1}\in\mathbb{R}^d$.

Two facts that are frequently used in the paper. Firstly, for a filtered probability space $(M,\Sigma,\mathbb{Q},(\mathcal{G}_t)_{t\in[r,R]})$, $1\le q\le\infty$, $r\le t\le R$ and $\xi\in L_q$, one has that

$$\|\xi-\mathbb{E}(\xi|\mathcal{G}_t)\|_q\le\sup_{t\le s\le R}\|\xi-\mathbb{E}(\xi|\mathcal{G}_s)\|_q\le 2\|\xi-\mathbb{E}(\xi|\mathcal{G}_t)\|_q \tag{5}$$

as a consequence that $\mathbb{E}(\cdot|\mathcal{F}_s)$ is a contraction on L_q. Secondly, given the assumptions on our forward diffusion, a polynomially bounded Borel function $g:\mathbb{R}^d\to\mathbb{R}$, $r\le t\le R\le T$ and $1\le q<\infty$, we have that

$$\begin{aligned}&\|g(X_R^{r,x})-\mathbb{E}(g(X_R^{r,x})|\mathcal{F}_t^r)\|_q\\&\le\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|g(\xi)-g(\eta)|^q\Gamma(r,x;t,y)\Gamma(t,y;R,\xi)\Gamma(t,y;R,\eta)dyd\xi d\eta\right)^{\frac{1}{q}}\\&\le 2\left\|g(X_R^{r,x})-\mathbb{E}(g(X_R^{r,x})|\mathcal{F}_t^r)\right\|_q.\end{aligned} \tag{6}$$

1.2 Functional fractional smoothness

The usage of fractional smoothness in the investigation of variational properties of BSDEs is the central idea of this paper. Fractional smoothness can be defined in various ways. One way is the so-called K-method, a method where functions are decomposed into differentiable parts and parts that are not differentiable. A quantitative analysis of these decompositions leads to fractional smoothness.

To be more precise, assume two Banach spaces X_0 and X_1, where (say) X_1 is continuously embedded into X_0, $0<t<\infty$ and $x\in X_0$, and recall that the K-functional is given by

$$K(x,t;X_0,X_1):=\inf\{\|x_0\|_{X_0}+t\|x_1\|_{X_1}:x=x_0+x_1\}.$$

For $0<\theta<1$ and $1\le q\le\infty$ this leads to the real interpolation spaces

$$\|x\|_{(X_0,X_1)_{\theta,q}}:=\left\|t^{-\theta}K(x,t;X_0,X_1)\right\|_{L_q\left((0,\infty),\frac{dt}{t}\right)}$$

with

$$X_1\subseteq(X_0,X_1)_{\theta_1,q_1'}\subseteq(X_0,X_1)_{\theta_1,q_1}\subseteq(X_0,X_1)_{\theta_0,q_0}\subseteq X_0$$

where $0 < \theta_0 < \theta_1 < 1$ and $1 \le q_0, q_1, q_1' \le \infty$ with $q_1' \le q_1$ (see [2, 3]). Applying this concept to the Malliavin Sobolev space $D_{1,p}$, we obtain the Malliavin Besov (or fractional Sobolev) spaces

$$B_{p,q}^{\theta} := (L_p, D_{1,p})_{\theta,q} \tag{7}$$

where $0 < \theta < 1$ is the main parameter of the smoothness and $1 < q \le \infty$ the fine-tuning parameter. In a context close to this paper these spaces and related ones have been exploited for example in [10, 11, 14, 21]. The classical setting of the Wiener space is changed in [10, 14] into a setting where the standard Gaussian measure is replaced by the distribution of the forward diffusion. Here we go one step ahead and replace $0 < \theta < 1$ by a vector $\Theta = (\theta_1, ..., \theta_L)$, where θ_l describes the smoothness at time r_l:

Definition 1.2. Let $\Theta = (\theta_1, ..., \theta_L) \in (0,1]^L$, $2 \le p < \infty$ and $\xi \in L_p$. If Y is the solution of the BSDE with generator f and terminal condition ξ, then we let $(\xi, f) \in B_{p,\infty}^{\Theta}(X)$ provided that there is some $c > 0$ such that

$$\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p \le c(r_l - s)^{\frac{\theta_l}{2}}$$

for all $l = 1, ..., L$ and $r_{l-1} \le s < r_l$. The infimum over all possible $c > 0$ is denoted by

$$c_{B_{p,\infty}^{\Theta}} = c_{B_{p,\infty}^{\Theta}}(\xi, f).$$

In the case that $f = 0$ we will simply write $\xi \in B_{p,\infty}^{\Theta}(X)$.

Specializing to $p = 2$ and to the linear one-step Gaussian case ($X = W$, $T = L = 1$ and $f = 0$) it holds (see [11, Corollary 2.3]) that

$$g(W_1) \in B_{2,\infty}^{(\theta)}(W) \quad \text{if and only if} \quad g \in B_{2,\infty}^{\theta}(\mathbb{R}^d, \gamma_d),$$

where the Wiener space over the standard Gaussian measure γ_d on $\mathbb{R}^d$ is considered. In particular, for $d = 1$ and for the orthonormal basis consisting of Hermite polynomials $(h_k)_{k=0}^{\infty} \subseteq L_2(\mathbb{R}, \gamma_1)$ we obtain that $g = \sum_{k=0}^{\infty} \alpha_k h_k \in B_{2,\infty}^{\theta}(\mathbb{R}, \gamma_1)$ if and only if there is some $c > 0$ such that for all $0 \le t < 1$ one has that

$$\sum_{k=1}^{\infty} k t^{k-1} \alpha_k^2 \le \frac{c^2}{(1-t)^{1-\theta}},$$

see [11, Theorem 2.2]. These connections explain the notation (p, ∞) in Definition 1.2. For a more general connection between the speed of convergence of the conditional expectations used in Definition 1.2 and the real interpolation method the reader is referred to [11]. Our definition of fractional smoothness by an upper bound of

$$\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p$$

has the advantage that (2) and (3) give the upper bounds

$$\left\|\nabla_x F_l(\overline{X}_{l-1}; s, X_s)\right\|_p \le \kappa_{p'} \frac{\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p}{\sqrt{r_l - s}} \le \kappa_{p'} c_{B^{\Theta}_{p,\infty}} (r_l - s)^{\frac{\theta_l - 1}{2}}$$

and

$$\left\|D^2 F_l(\overline{X}_{l-1}; s, X_s)\right\|_p \le \kappa_{p'} \frac{\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p}{r_l - s} \le \kappa_{p'} c_{B^{\Theta}_{p,\infty}} (r_l - s)^{\frac{\theta_l - 2}{2}}$$

for $r_{l-1} \le s < r_l$ and $1 = (1/p) + (1/p')$, so that we can control the gradient and the Hessian of F_l. For our paper the fine-tuning parameter $q = \infty$ in (the generalization of) (7) turns out to be the right one.

Finally, we want to mention the coincidence, that most of the relevant examples are naturally linked to this fine-tuning parameter $q = \infty$ in (7).

1.3 Time-nets, splines and entropy numbers

In our BSDE system the Z-process gets possibly singular at any of the particular time points r_l when r_l is approached from the left. The degree of this singularity is determined by the parameter θ_l describing the fractional smoothness in r_l. To keep the variation $\mathrm{var}_p(g(X_{r_1}, ..., X_{r_L}), f, \tau)$ small, we have to choose time-nets which refine on the left of r_l with an order given by the fractional smoothness θ_l while each of the intervals $[r_{l-1}, r_l]$ is divided into n sub-intervals.

Definition 1.3. For $\Theta \in (0,1]^L$ we let $\tau^{n,\Theta} = (t_k^{n,\Theta})_{k=0}^{nL}$ be given by $t_0^{n,\Theta} := 0$ and

$$t_k^{n,\Theta} := r_{l-1} + (r_l - r_{l-1}) \left(1 - \left(1 - \frac{k - (l-1)n}{n}\right)^{\frac{1}{\theta_l}}\right) \quad \text{for} \quad (l-1)n < k \le ln.$$

Estimates on the L_p-variation $\|Y_t - Y_s\|_p$ are close to estimates how good the process Y can be approximated in L_p by linear adapted splines, i.e. we simply compute adapted approximations of Y at the time-points $t_0, ..., t_n$ and interpolate them linearly. So the notion *adapted spline* refers to the fact that the knots are adapted, however the spline itself is not an adapted process. The adapted splines are typically used in complexity theory for stochastic processes to find efficient approximation schemes for stochastic processes where the whole path needs to be approximated but the adaptedness of the approximation is not fully needed, see [7]. Here we use the following notation:

Definition 1.4. Given a time-net $\tau = (t_k)_{k=0}^n$ with $r = t_0 < \cdots < t_n = R \le T$ we say that the process $S = (S_t)_{t \in [r,R]}$ is an adapted spline based on τ provided that S_{t_k} is $\mathcal{F}_{t_k}$-measurable for all $k = 0, ..., n$ and

$$S_t := \frac{t_k - t}{t_k - t_{k-1}} S_{t_{k-1}} + \frac{t - t_{k-1}}{t_k - t_{k-1}} S_{t_k} \quad \text{for} \quad t_{k-1} \le t \le t_k.$$

Finally, we recall the notion of entropy numbers to measure and compare compactness properties of $Y = (Y_s)_{s\in[t,r_l]}$ as $t \uparrow r_l$ where the process gets singular.

Definition 1.5. Given a normed space E and $A \subseteq E$ we define $e_n(A|E) := \inf \varepsilon$, where the infimum is taken over all $\varepsilon > 0$ such that there are $x_1, ..., x_n \in E$ with

$$A \subseteq \bigcup_{i=1}^{n} \{x_i + \varepsilon B_E\} \quad \text{with} \quad B_E := \{x \in E : \|x\| \le 1\}.$$

2 Functional fractional smoothness and BSDEs

2.1 A general equivalence

The basic result of this paper is

Theorem 2.1. *Assume that $(A_{b,\sigma})$, (A_f) and (A_g) are satisfied. For $2 \le p < \infty$ and* fixed *$l \in \{1, ..., L\}$ and $\theta_l \in (0, 1]$ consider the following conditions:*

$(C1_l)$ *There is some $c_1 > 0$ such that, for $r_{l-1} \le s < t < r_l$,*

$$\|Z_t - Z_s\|_p \le c_1 \left(\int_s^t (r_l - r)^{\theta_l - 2} dr \right)^{\frac{1}{2}}.$$

$(C2_l)$ *There is some $c_2 > 0$ with $\|Z_t\|_p \le c_2 (r_l - t)^{\frac{\theta_l - 1}{2}}$ for $r_{l-1} \le t < r_l$.*

$(C3_l)$ *There is some $c_3 > 0$ such that, for $r_{l-1} \le s < t \le r_l$,*

$$\|Y_t - Y_s\|_p \le c_3 \left(\int_s^t (r_l - r)^{\theta_l - 1} dr \right)^{\frac{1}{2}}.$$

$(C4_l)$ *There is some $c_4 > 0$ such that, for $r_{l-1} \le s < r_l$,*

$$\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p \le c_4 (r_l - s)^{\frac{\theta_l}{2}}.$$

$(C5_l)$ *There is some $c_5 > 0$ such that, for $r_{l-1} \le t < r_l$,*

$$\left\| \left(\int_{r_{l-1}}^t |(D^2 F_l)(\overline{X}_{l-1}; s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_p \le c_5 (r_l - t)^{\frac{\theta_l - 1}{2}}.$$

$(C6_l)$ *There is some $c_6 > 0$ such that for all $n = 1, 2, ...$ there is an adapted spline $S^n = (S^n_t)_{t\in[r_{l-1}, r_l]}$ based on*

$$\left(r_{l-1} + (r_l - r_{l-1}) \left(1 - \left(1 - \frac{k}{n} \right)^{\frac{1}{\theta_l}} \right) \right)_{k=0}^{n}$$

such that

$$\sqrt{n} \sup_{t\in[r_{l-1}, r_l]} \|Y_t - S^n_t\|_p \le c_6.$$

The spline can be arranged such that $S^n_{r_{l-1}} = Y_{r_{l-1}}$ and $S^n_{r_l} = Y_{r_l}$.

($C7_l$) *There is some* $c_7 > 0$ *such that for* $r_{l-1} \le t < r_l$ *one has that*

$$\sup_{n\ge 1} \sqrt{n} e_n\big((Y_s)_{s\in[t,r_l]}|L_p\big) \le c_7(r_l - t)^{\frac{\theta_l}{2}}.$$

Then one has that

$$(C1_l) \overset{\theta_l\in(0,1)}{\Longrightarrow} (C2_l) \Longleftrightarrow (C3_l) \Longleftrightarrow$$
$$(C4_l) \Longleftrightarrow (C5_l) \Longleftrightarrow (C6_l) \Longleftrightarrow (C7_l) \Longrightarrow (C1_l).$$

Remark 2.2. The implication $(C1_l) \Longrightarrow (C2_l)$ does not hold in general. To see this we consider $d = T = L = l = 1$, $f = 0$, $\theta_1 = 1$ and $p = 2$, and let

$$g = \sum_{n=0}^{\infty} \alpha_n h_n \quad \text{with} \quad \sum_{n=0}^{\infty} \alpha_n^2 < \infty$$

where $(h_n)_{n=0}^{\infty} \subseteq L_2(\mathbb{R}, \gamma_1)$ is the orthonormal basis of Hermite polynomials. Then, as in [11, Lemma 3.9], we get that

$$\left\| \frac{\partial^2 F_1}{\partial x^2}(t, W_t) \right\|_2^2 = \sum_{n=0}^{\infty} \alpha_{n+2}^2 (n+2)(n+1) t^n$$

and

$$\|Z_t - Z_s\|_2^2 = \int_s^t \sum_{n=0}^{\infty} \alpha_{n+2}^2 (n+2)(n+1) r^n dr.$$

Choosing $\alpha_n := (n(n-1))^{-1/2}$ for $n \ge 2$ and $\alpha_0 = \alpha_1 = 0$ gives $(C1_l)$ but $\sup_{0\le t<1} \|Z_t\|_2 = \infty$.

From Theorem 2.1 the multi-step case directly follows. For its formulation we introduce for $\Theta = (\theta_1, ..., \theta_L) \in (0, 1]^L$ and $0 \le t < T$ the function

$$\varphi(t) := \sum_{l=1}^{L} \chi_{[r_{l-1}, r_l)}(t)(r_l - t)^{\frac{\theta_l - 1}{2}}.$$

Theorem 2.3. *Assume that* $(A_{b,\sigma})$, (A_f) *and* (A_g) *are satisfied. For* $2 \le p < \infty$ *and* $\Theta \in (0, 1]^L$ *consider the following conditions:*

(C1) *There is some* $c_1 > 0$ *such that, for* $r_{l-1} \le s < t < r_l$,

$$\|Z_t - Z_s\|_p \le c_1 \left(\int_s^t \frac{\varphi(r)^2}{r_l - r} dr \right)^{\frac{1}{2}}.$$

(C2) *There is some* $c_2 > 0$ *with* $\|Z_t\|_p \le c_2 \varphi(t)$ *for* $0 \le t < T$.

(C3) *There is some $c_3 > 0$ such that, for $r_{l-1} \leq s < t \leq r_l$,*

$$\|Y_t - Y_s\|_p \leq c_3 \left(\int_s^t \varphi(r)^2 dr \right)^{\frac{1}{2}}.$$

(C4) $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$.

(C6) *There is some $c_6 > 0$ such that for all $n = 1, 2, ...$ there is an adapted spline $S^n = (S^n_t)_{t\in[0,T]}$ based on $\tau^{n,\Theta}$ such that*

$$\sqrt{n} \sup_{t\in[0,T]} \|Y_t - S^n_t\|_p \leq c_6.$$

Then one has that

$$(C1) \stackrel{\Theta\in(0,1)^L}{\Longrightarrow} (C2) \Longleftrightarrow (C3) \Longleftrightarrow (C4) \Longleftrightarrow (C6) \Longrightarrow (C1).$$

The remaining properties $(C5_l)$ and $C7_l)$ could be included as well. By using the properties (C3) and (C1) we deduce by a simple computation

Corollary 2.4. *For $0 < \theta_l' < \theta_l < 1$, $l = 1, ..., L$ and $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$ one has that*

$$\sup_n \sqrt{n}\, \mathrm{var}_p(\xi, f, \tau^{n,\Theta'}) < \infty.$$

Examples will be considered in Example 2.9 and Theorem 2.10. The proof of Theorem 2.1 is postponed to Section 3.1.

2.2 Sufficient conditions for fractional smoothness

In this section we describe sufficient conditions on ξ for the condition $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$ which are *independent from the generator f*. Note that in the case $L = 1$ it follows by definition that $(\xi, 0) \in B^{\Theta}_{p,\infty}(X)$ implies that $(\xi, f) \in B^{\Theta}_{p,\infty}(X)$. To our knowledge it is open whether it still holds for $L > 1$.

2.2.1 The first sufficient condition

The first sufficient condition is based on the concept to measure the fractional smoothness of a random variable on the Wiener space by mixing the underlying Gaussian structure with an independent copy and to look how sensitive the given random variable is with respect to this operation (see, for example, [18]). In our setting this would correspond to comparing, for example, $g(X_1)$ with $g(X_1^\eta)$ where X_1^η is defined via a Brownian motion $W_t^\eta := \sqrt{1-\eta^2}W_t + \eta B_t$ with B being a Brownian motion independent from W and $0 \leq \eta \leq 1$. Because we have a time-dependent structure we extend this concept by allowing more general operations with W and its independent copy B.

Let us consider two independent d-dimensional Brownian motions W and B on the same complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ starting in zero, and let us denote by $(\mathcal{F}_t^W)_{t\in[0,T]}$ (resp. $(\mathcal{F}_t^B)_{t\in[0,T]}$ and $(\mathcal{F}_t^{W,B})_{t\in[0,T]}$) the $\mathbb{P}$-augmentation of the natural filtrations of W (resp. B and (W,B)). For a measurable function $\eta : [0,T] \mapsto [-1,1]$ we define the standard d-dimensional $\mathcal{F}^{W,B}$-Brownian motion

$$W_t^\eta := \int_0^t \sqrt{1-\eta(s)^2}dW_s + \int_0^t \eta(s)dB_s$$

and denote by $(\mathcal{F}_t^\eta)_{t\in[0,T]}$ the augmentation of its natural filtration. We also define X^η to be the strong $(\mathcal{F}_t^\eta)_{t\in[0,T]}$-measurable solution of

$$X_t^\eta = x_0 + \int_0^t b(s, X_s^\eta)ds + \int_0^t \sigma(s, X_s^\eta)dW_s^\eta.$$

For a given $\mathcal{F}_T^\eta$-measurable terminal condition $\xi^\eta \in L_p$ with $2 \le p < \infty$ we let (Y^η, Z^η) be the L_p-solution in the filtration $(\mathcal{F}_t^\eta)_{t\in[0,T]}$ of

$$Y_t^\eta = \xi^\eta + \int_t^T f(s, X_s^\eta, Y_s^\eta, Z_s^\eta)ds - \int_t^T Z_s^\eta dW_s^\eta.$$

In the case $\eta \equiv 0$ we simply write $W = W^0$, $\xi = \xi^0$, $(X,Y,Z) = (X^0, Y^0, Z^0)$, and $\mathcal{F}_t = \mathcal{F}_t^0$. Our aim is to bound the distance between (X^η, Y^η, Z^η) and (X,Y,Z) by the following stability result:

Theorem 2.5. *Assume that $(A_{b,\sigma})$ and (A_f) are satisfied. Then for $2 \le p < \infty$ and $\xi, \xi^\eta \in L_p$ we have that*

$$\left\| \sup_{0\le t\le T} |X_t^\eta - X_t| \right\|_p + \left\| \sup_{0\le t\le T} |Y_t^\eta - Y_t| \right\|_p + \left\| \left(\int_0^T |Z_t^\eta - Z_t|^2 dt \right)^{1/2} \right\|_p$$
$$\le c\left[\|\xi^\eta - \xi\|_p + [1 + \|\xi\|_p] \sqrt{\int_0^T \eta(t)^2 dt} \right]$$

where $c > 0$ depends at most on $(p, T, b, \sigma, K_f, L_f)$ and is non-decreasing with respect to K_f and L_f.

The proof can be found in Section 3.2. The motivation for the result is Corollary 2.6 below. To formulate it, given $0 \le t < r \le T$ we let

$$\eta_{t,r}(s) := \chi_{(t,r]}(s),$$

i.e. we replace the Brownian paths on $(t,r]$ by an independent copy.

Corollary 2.6. *Assume $2 \le p < \infty$, $(A_{b,\sigma})$, (A_f) and $\xi = g(X_{r_1}, ..., X_{r_L}) \in L_p$ for some Borel measurable function $g : \mathbb{R}^L \to \mathbb{R}$. Let*

$$\xi^{t,r} := g(X_{r_1}^{\eta_{t,r}}, ..., X_{r_L}^{\eta_{t,r}})$$

for $0 \le t < r \le T$ and let $\Theta = (\theta_1, ..., \theta_L) \in (0,1]^L$. If there is a constant $c > 0$ such that one has that

$$\|\xi - \xi^{t,r_l}\|_p \le c(r_l - t)^{\frac{\theta_l}{2}} \tag{8}$$

for all $l = 1, ..., L$ and $r_{l-1} \le t < r_l$, then $(\xi, f) \in B_{p,\infty}^{\Theta}(X)$.

Proof. For $r_{l-1} \le t < r_l$ we get by (6) that

$$\begin{aligned} \|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_t)\|_p &\le \|Y_{r_l} - Y_{r_l}^{\eta_{t,r_l}}\|_p \\ &\le c_{(2.5)} \left[\|\xi - \xi^{t,r_l}\|_p + [1 + \|\xi\|_p] \sqrt{\int_0^T \eta_{t,r_l}(r)^2 dr} \right] \\ &\le c_{(2.5)} \left[c(r_l - t)^{\frac{\theta_l}{2}} + [1 + \|\xi\|_p] \sqrt{r_l - t} \right]. \end{aligned}$$

□

Using a truncation argument, we obtain a modified version of Theorem 2.3, without assuming that g is polynomially bounded nor that f is continuously differentiable in (x, y, z).

Corollary 2.7. *Assume $(A_{b,\sigma})$ and that the generator $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuous in (t, x, y, z) and that there is some $L_f > 0$ such that*

$$|f(s, x_1, y_1, z_1) - f(s, x_2, y_2, z_2)| \le L_f[|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|].$$

Let $2 \le p < \infty$, $\xi = g(X_{r_1}, ..., X_{r_L}) \in L_p$ for some Borel measurable function $g : \mathbb{R}^L \to \mathbb{R}$, $\Theta \in (0,1]^L$ and let (Y, Z) be the L_p-solution of our BSDE. Assume that condition (8) is satisfied. Then there are sets $\mathcal{N}_l \subseteq [r_{l-1}, r_l)$ of Lebesgue measure zero such that the following is satisfied:

(C1') *There is some $c_1 > 0$ such that for $s, t \in [r_{l-1}, r_l) \setminus \mathcal{N}_l$ with $r_{l-1} \le s < t < r_l$ one has*

$$\|Z_t - Z_s\|_p \le c_1 \left(\int_s^t \frac{\varphi(t)^2}{r_l - r} dr \right)^{\frac{1}{2}}.$$

(C2') *There is some $c_2 > 0$ with $\|Z_t\|_p \le c_2 \varphi(t)$ for $t \in \bigcup_{l=1}^L ([r_{l-1}, r_l) \setminus \mathcal{N}_l)$.*

(C3') *There is some $c_3 > 0$ such that, for $r_{l-1} \le s < t \le r_l$, one has*

$$\|Y_t - Y_s\|_p \le c_3 \left(\int_s^t \varphi(r)^2 dr \right)^{\frac{1}{2}}.$$

Proof. (a) Let $(f^N)_{N\ge 1}$ be a sequence of generators satisfying assumption (A_f) such that

(i) $\lim_N \left\| \int_0^T |f^N(s,X_s,Y_s,Z_s) - f(s,X_s,Y_s,Z_s)| ds \right\|_p = 0$,

(ii) $K_{f^N} \le 2K_f$ and $L_{f^N} \le L_f$.

(b) Letting $y^N = -N \vee y \wedge N$ for $y \in \mathbb{R}$ and $N \ge 1$, ξ^N satisfies (A_g) and $\|\xi^N - \xi\|_p \to 0$ as $N \to \infty$. In addition, for all $l = 1, ..., L$ and $r_{l-1} \le t < r_l$ we have

$$\|\xi^N - (\xi^N)^{t,r_l}\|_p = \|\xi^N - (\xi^{t,r_l})^N\|_p \le \|\xi - \xi^{t,r_l}\|_p \le c_{(8)}(r_l - t)^{\frac{\theta_l}{2}}.$$

(c) To (ξ^N, f^N) we associate (Y^N, Z^N) as BSDE solution in L_p. In view of the inequality above and according to Corollary 2.6, $(\xi^N, f^N) \in B^\Theta_{p,\infty}(X)$. Because K_{f^N}, L_{f^N} and $\|\xi^N\|_p$ are bounded independently of N, we have

$$\sup_{N\ge 1} c_{B^\Theta_{p,\infty}}(\xi^N, f^N) < \infty,$$

which follows by the proof of Corollary 2.6. Theorem 2.3 applies to (Y^N, Z^N) for each N and there are $c^N > 0$ such that

$$\|Z_{N,t} - Z_{N,s}\|_p \le c^N \left(\int_s^t \frac{\varphi(t)^2}{r_l - r} dr \right)^{\frac{1}{2}}$$

for $r_{l-1} \le s < t < r_l$. Looking at the constants in the proof of $(C4_l) \Rightarrow (C1_l)$ we realize that we can take $\sup_N c^N =: c < \infty$. By Lemma A.1 applied to $\xi^{(0)} = \xi$, $f_0(\omega,s,y,z) := f(s,X_s(\omega),y,z)$, $(Y^{(0)}, Z^{(0)}) = (Y,Z)$, and $\xi^{(1)} = \xi^N$, $f_1(\omega,s,y,z) := f^N(s,X_s(\omega),y,z)$, $(Y^{(1)}, Z^{(1)}) = (Y^N, Z^N)$, there is a subsequence $(N_k)_{k=1}^\infty$ such that $Z_{N_k,t}$ converges to Z_t a.s. for $t \in [r_{l-1}, r_l) \setminus \mathcal{N}_l$ for some $\mathcal{N}_l$ of Lebesgue measure zero. Fatou's lemma gives

$$\|Z_t - Z_s\|_p \le c \left(\int_s^t \frac{\varphi(t)^2}{r_l - r} dr \right)^{\frac{1}{2}}$$

for $r_{l-1} \le s < t < r_l$ with $s,t \in [r_{l-1}, r_l) \setminus \mathcal{N}_l$. As in the proof of $(C1_l) \Longrightarrow (C2_l) \Longrightarrow (C3_l)$ below we can deduce $(C2')$ and $C3')$ where in the case $r_{l-1} \in \mathcal{N}_l$ in $(C1_l) \Longrightarrow (C2_l)$ we have to replace $\|Z_{r_{l-1}}\|_p$ by $\liminf_n \|Z_{\rho_n}\|_p$ with $\rho_n \in [r_{l-1}, r_l) \setminus \mathcal{N}_l$ and $\rho_n \downarrow r_{l-1}$ $\square$

Definition 2.8. *A measurable function $g : \mathbb{R} \to \mathbb{R}$ is of bounded variation, in short $g \in \mathrm{BV}$, provided that*

$$\|g\|_{BV} := \sup_N \sup_{-\infty < x_0 < \cdots < x_N < \infty} \sum_{k=1}^N |g(x_k) - g(x_{k-1})| < \infty.$$

The following Example 2.9 is more general than needed in this paper, however this generality does not require any extra effort and constitutes the natural setting.

Example 2.9. *Assume $0 < \theta < \frac{1}{p} \le \alpha \le 1$, $g_j \in \mathrm{BV}$ with $\sum_{j=1}^{\infty} \|g_j\|_{BV}^{\alpha} < \infty$, and linear and continuous functionals $\mu_1, \mu_2, ... \in (C[0,T])^*$ with $\|\mu_j\| \le 1$ such that the laws of $\langle X, \mu_1\rangle$, $\langle X, \mu_2\rangle$, $\langle X, \mu_3\rangle$, ... have densities bounded uniformly by a constant $\beta > 0$. Define*

$$\xi := \Phi(g_1(\langle X, \mu_1\rangle), g_2(\langle X, \mu_2\rangle), ...),$$

where Φ is a measurable function such that

$$|\Phi(x_1, x_2, ...) - \Phi(y_1, y_2, ...)| \le \kappa \sum_{j=1}^{\infty} |x_j - y_j|^{\alpha}$$

for some $\kappa > 0$. Then there is a constant $c > 0$ such that for all measurable $\eta : [0,T] \to [-1,1]$ we have that

$$\|\xi - \xi^{\eta}\|_p \le c \left(\int_0^T \eta(r)^2 dr \right)^{\frac{\theta}{2}}.$$

Consequently, given $\Theta \in (0, 1/p)^L$ there is a constant $c' > 0$ such that

$$\|\xi - \xi^{t,r_l}\|_p \le c'(r_l - t)^{\frac{\theta_l}{2}}$$

for $r_{l-1} \le t < r_l$.

Proof. Using [1, Theorem 2.4] for $1 \le q < \infty$ we get that

$$\begin{aligned}
\|\xi - \xi^{\eta}\|_p &\le \kappa \left\| \sum_{j=1}^{\infty} |g_j(\langle X, \mu_j\rangle) - g_j(\langle X^{\eta}, \mu_j\rangle)|^{\alpha} \right\|_p \\
&\le \kappa \sum_{j=1}^{\infty} \|g_j(\langle X, \mu_j\rangle) - g_j(\langle X^{\eta}, \mu_j\rangle)\|_{\alpha p}^{\alpha} \\
&\le \kappa 3^{\alpha + \frac{1}{p}} \beta^{\frac{q}{q+1}\frac{1}{p}} \sum_{j=1}^{\infty} \|g_j\|_{BV}^{\alpha} \, \|\langle X, \mu_j\rangle - \langle X^{\eta}, \mu_j\rangle\|_q^{\frac{q}{q+1}\frac{1}{p}} \\
&\le \kappa 3^{\alpha + \frac{1}{p}} \beta^{\frac{q}{q+1}\frac{1}{p}} \sum_{j=1}^{\infty} \|g_j\|_{BV}^{\alpha} \sup_j \|\langle X, \mu_j\rangle - \langle X^{\eta}, \mu_j\rangle\|_q^{\frac{q}{q+1}\frac{1}{p}} \\
&\le \kappa 3^{\alpha + \frac{1}{p}} \beta^{\frac{q}{q+1}\frac{1}{p}} \left[\sum_{j=1}^{\infty} \|g_j\|_{BV}^{\alpha} \right] \left\| \sup_{0 \le t \le T} |X_t - X_t^{\eta}| \right\|_q^{\frac{q}{q+1}\frac{1}{p}} \\
&\le \kappa 3^{\alpha + \frac{1}{p}} \beta^{\frac{q}{q+1}\frac{1}{p}} \left[\sum_{j=1}^{\infty} \|g_j\|_{BV}^{\alpha} \right] \left(c_{(21)} \left(\int_0^T \eta(r)^2 dr \right)^{\frac{1}{2}} \right)^{\frac{q}{q+1}\frac{1}{p}},
\end{aligned}$$

where inequality (21) below is used. Taking $1 \le q < \infty$ large enough the assertion follows. $\square$

2.2.2 The second sufficient condition

The second sufficient condition relies on a simple iteration procedure:

Theorem 2.10. *Assume that $(A_{b,\sigma})$ and (A_f) are satisfied and that*

$$\xi := g(X_{r_1}, ..., X_{r_L}),$$

where

$$|g(x_1, ..., x_L) - g(x'_1, ..., x'_L)| \leq \sum_{l=1}^{L} [|g_l(x_l) - g_l(x'_l)| + \psi_l(x_1, ..., x_l; x'_1, ..., x'_l)|x_l - x'_l|]$$

with polynomially bounded Borel functions g, g_l and ψ_l such that

$$\|g_l(X_{r_l}) - \mathbb{E}(g_l(X_{r_l})|\mathcal{F}_t)\|_p \leq c(r_l - t)^{\frac{\theta_l}{2}} \tag{9}$$

for $l = 1, ..., L$, $0 < \theta_l \leq 1$, and $r_{l-1} \leq t < r_l$. Then,

$$(\xi, f) \in B^{\Theta}_{p,\infty}(X).$$

The proof of Theorem 2.10 is given in Section 3.3.

Example 2.11. Let $\Phi : \mathbb{R}^L \to \mathbb{R}$ be Lipschitz and $g_1, ..., g_L$ be as in Theorem 2.10, and define

$$g(x_1, ..., x_L) := \Phi(g_1(x_1), ..., g_L(x_L)).$$

To verify (9) for concrete functions g_l, it is sufficient to check the inequality for the Brownian motion and for an appropriately rescaled function:

Proposition 2.12. *Let $c_{(B.1)} > 0$ be the constant from Proposition B.1 so that*

$$\Gamma(t, x; s, \xi) \leq c_{(B.1)}\ \gamma^d_{s-t}\left(\frac{x - \xi}{c_{(B.1)}}\right)$$

and let $h_l(x) := g_l\left(x_0 + c_{(B.1)}x\right)$ and assume that

$$\|h_l(W_{r_l}) - \mathbb{E}(h_l(W_{r_l})|\mathcal{F}_t)\|_p \leq c_l(r_l - t)^{\frac{\theta_l}{2}} \quad \textit{for} \quad 0 \leq t < r_l, \tag{10}$$

then (9) holds true for some $c > 0$.

The proof of this proposition can be found in the appendix. One can rescale the argument of the function h_l in (10) as well to assume that $r_l = 1$. Examples for (10) with $d = 1$ and $r_l = 1$ are the following:

(a) If $h_l(x) = \chi_{[K,\infty)}(x)$ for some $K \in \mathbb{R}$, then $\theta = 1/p$ according to [12, Example 4.7, Proposition 4.5].

(b) If $h_l(x) = x^\alpha$ for $x \geq 0$ and $h(x) = 0$ otherwise, and $0 < \alpha < 1 - (1/p)$, then $\theta = \alpha + (1/p)$ according to [22, Example 5.2, Lemma 4.7] and [12, Proposition 4.5].

A precise investigation about the relation of (10) to $B^\theta_{p,q}(\mathbb{R}^d, \gamma_d)$ can be found in [13].

3 Proofs of the main results

3.1 Proof of Theorem 2.1

$(C1_l) \Longrightarrow (C2_l)$ for $0 < \theta_l < 1$ is obvious as

$$\begin{aligned}
\|Z_t\|_p &\leq \|Z_{r_{l-1}}\|_p + c_1 \left(\int_{r_{l-1}}^t (r_l - r)^{\theta_l - 2} dr \right)^{\frac{1}{2}} \\
&= \|Z_{r_{l-1}}\|_p + c_1 \left(\frac{1}{1-\theta_l} [(r_l - t)^{\theta_l - 1} - (r_l - r_{l-1})^{\theta_l - 1}] \right)^{\frac{1}{2}} \\
&\leq \|Z_{r_{l-1}}\|_p + c_1 (1-\theta_l)^{-\frac{1}{2}} (r_l - t)^{\frac{\theta_l - 1}{2}}.
\end{aligned}$$

$(C2_l) \Longrightarrow (C3_l)$ We observe that

$$\begin{aligned}
& \|Y_t - Y_s\|_p \\
= & \left\| \int_s^t f(r, X_r, Y_r, Z_r) dr - \int_s^t Z_r dW_r \right\|_p \\
\leq & \int_s^t \|f(r, X_r, Y_r, Z_r)\|_p dr + a_p \left(\int_s^t \|Z_r\|_p^2 dr \right)^{\frac{1}{2}} \\
\leq & K_f (t-s) + L_f \int_s^t \||X_r| + |Y_r| + |Z_r|\|_p dr + a_p \left(\int_s^t \|Z_r\|_p^2 dr \right)^{\frac{1}{2}} \\
\leq & (t-s) \left[K_f + L_f \sup_{r \in [0,T]} \|X_r\|_p + L_f \sup_{r \in [0,T]} \|Y_r\|_p \right] \\
& + c_2 (L_f \sqrt{T} + a_p) \left(\int_s^t (r_l - r)^{\theta_l - 1} dr \right)^{\frac{1}{2}}
\end{aligned}$$

where we used that $2 \leq p < \infty$ and where $a_p > 0$ is the constant from the Burkholder-Davis-Gundy inequality.

$(C3_l) \Longrightarrow (C4_l)$ Here we get that

$$\begin{aligned}
\|Y_{r_l} - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p &\leq \|Y_{r_l} - Y_s\|_p + \|Y_s - \mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p \\
&\leq 2\|Y_{r_l} - Y_s\|_p \\
&\leq 2c_3 \left(\int_s^{r_l} (r_l - r)^{\theta_l - 1} dr \right)^{\frac{1}{2}} \\
&= 2c_3 \sqrt{\frac{1}{\theta_l}} (r_l - s)^{\frac{\theta_l}{2}}.
\end{aligned}$$

$(C4_l) \Longrightarrow (C5_l)$ We consider

$$\left\| \left(\int_{r_{l-1}}^t |(D^2 F_l)(\overline{X}_{l-1}; s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_p$$

$$
\begin{aligned}
&= \left\| \left(\sum_{k=1}^{d} \int_{r_{l-1}}^{t} |(\nabla_x(\partial_{x_k} F_l))(\overline{X}_{l-1}; s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_p \\
&\leq \frac{1}{\eta} \left\| \left(\sum_{k=1}^{d} \int_{r_{l-1}}^{t} |(\nabla_x(\partial_{x_k} F_l)\sigma)(\overline{X}_{l-1}; s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_p \\
&\leq \sum_{k=1}^{d} \frac{1}{\eta} \left\| \left(\int_{r_{l-1}}^{t} |(\nabla_x(\partial_{x_k} F_l)\sigma)(\overline{X}_{l-1}; s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_p \\
&\leq \sum_{k=1}^{d} \frac{b_p}{\eta} \left\| \int_{r_{l-1}}^{t} (\nabla_x(\partial_{x_k} F_l)\sigma)(\overline{X}_{l-1}; s, X_s) dW_s \right\|_p
\end{aligned}
$$

where $b_p > 0$ is the constant from the Burkholder-Davis-Gundy inequality and the ellipticity condition on σ implies that there exists an $\eta > 0$ such that

$$
\eta |y|_{\mathbb{R}^d} \leq |y^* \sigma(t,x)|_{\mathbb{R}^d} \quad \text{for all} \quad x, y \in \mathbb{R}^d.
$$

To upper-bound the terms of the last sum we use Itô's formula and our PDE (which reduces the number of terms) to obtain

$$
\begin{aligned}
&\partial_{x_k} F_l(\overline{X}_{l-1}; t, X_t) - \partial_{x_k} F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}}) \\
= \; & -\int_{r_{l-1}}^{t} \left[\langle \partial_{x_k} b, \nabla_x F_l \rangle + \frac{1}{2} \langle \partial_{x_k} A, D^2 F_l \rangle \right] (\overline{X}_{l-1}; s, X_s) ds \qquad (11) \\
&+ \int_{r_{l-1}}^{t} (\nabla_x(\partial_{x_k} F_l)\sigma)\, (\overline{X}_{l-1}; s, X_s) dW_s
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left\| \int_{r_{l-1}}^{t} (\nabla_x(\partial_{x_k} F_l)\sigma)(\overline{X}_{l-1}; s, X_s) dW_s \right\|_p \\
\leq \; & \|\nabla_x F_l(\overline{X}_{l-1}; t, X_t)\|_p + \|\nabla_x F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}})\|_p \\
&+ \left\| \int_{r_{l-1}}^{t} \left[\langle \partial_{x_k} b, \nabla_x F_l \rangle + \frac{1}{2} \langle \partial_{x_k} A, D^2 F_l \rangle \right] (\overline{X}_{l-1}; s, X_s) ds \right\|_p \\
\leq \; & \kappa_{p'} \frac{R_t}{\sqrt{r_l - t}} + \kappa_{p'} \frac{R_{r_{l-1}}}{\sqrt{r_l - r_{l-1}}} + \kappa_{p'} \|\partial_{x_k} b\|_\infty \int_{r_{l-1}}^{r_l} \frac{R_s}{\sqrt{r_l - s}} ds \\
&+ \kappa_{p'} \frac{\|\partial_{x_k} A\|_\infty}{2} \int_{r_{l-1}}^{r_l} \frac{R_s}{r_l - s} ds
\end{aligned}
$$

with $R_s := \|Y_{r_l} - \mathbb{E}(Y_{r_l} | \mathcal{F}_s)\|_p$ and $r_{l-1} \leq s < r_l$ where we used $(A_{b,\sigma})$ and inequalities (2) and (3). Consequently,

$$
\left\| \left(\int_{r_{l-1}}^{t} |(D^2 F_l)(\overline{X}_{l-1}; s, X_s)|^2 ds \right)^{\frac{1}{2}} \right\|_p
$$

$$
\begin{aligned}
&\leq c_4 \frac{db_p}{\eta}\kappa_{p'}\Bigg[(r_l-t)^{\frac{\theta_l-1}{2}}+(r_l-r_{l-1})^{\frac{\theta_l-1}{2}}\\
&\quad+\sup_{1\leq k\leq d}\|\partial_{x_k}b\|_\infty\int_{r_{l-1}}^{r_l}(r_l-s)^{\frac{\theta_l-1}{2}}ds\\
&\quad+\sup_{1\leq k\leq d}\frac{\|\partial_{x_k}A\|_\infty}{2}\int_{r_{l-1}}^{r_l}(r_l-s)^{\frac{\theta_l}{2}-1}ds\Bigg].
\end{aligned}
$$

$\underline{(C5_l)\Longrightarrow(C2_l)}$ Here we start with

Lemma 3.1. *Assume that $(A_{b,\sigma}),(A_f)$ and (A_g) are satisfied. There exists a constant $c>0$, depending at most on σ,b,T,d and $2\leq p<\infty$, such that, for all $r_{l-1}\leq s<t<r_l$,*

$$
\begin{aligned}
&\|\nabla_x F_l(\overline{X}_{l-1};t,X_t)-\nabla_x F_l(\overline{X}_{l-1};s,X_s)\|_p\\
&\quad\leq c(t-s)\|\nabla_x F_l(\overline{X}_{l-1};r_{l-1},X_{r_{l-1}})\|_p\\
&\quad\quad+c(t-s)\left\|\left(\int_{r_{l-1}}^s|D^2F_l(\overline{X}_{l-1};v,X_v)|^2dv\right)^{\frac{1}{2}}\right\|_p\\
&\quad\quad+c\left\|\left(\int_s^t|D^2F_l(\overline{X}_{l-1};v,X_v)|^2dv\right)^{\frac{1}{2}}\right\|_p.
\end{aligned}
$$

Proof. For simplicity we will omit $\overline{X}_{l-1}$ in the computation. Using (11) with r_{l-1} replaced by s we get that

$$
\begin{aligned}
&\|\nabla_x F_l(t,X_t)-\nabla_x F_l(s,X_s)\|_p\\
&\leq \sum_{k=1}^d\|\partial_{x_k}F_l(t,X_t)-\partial_{x_k}F_l(s,X_s)\|_p\\
&\leq \left[\sum_{k=1}^d\|\partial_{x_k}b\|_\infty\right]\left\|\int_s^t|\nabla_x F_l(v,X_v)|dv\right\|_p\\
&\quad+\left[\sum_{k=1}^d\frac{\|\partial_{x_k}A\|_\infty}{2}\right]\left\|\int_s^t|D^2F_l(v,X_v)|dv\right\|_p\\
&\quad+a_p\sum_{k=1}^d\left\|\left(\int_s^t|(\nabla_x(\partial_{x_k}F_l)\sigma)(v,X_v)|^2\,dv\right)^{\frac{1}{2}}\right\|_p\\
&\leq \left[\sum_{k=1}^d\|\partial_{x_k}b\|_\infty\right]\left\|\int_s^t|\nabla_x F_l(v,X_v)|dv\right\|_p\\
&\quad+\left[\sum_{k=1}^d\frac{\|\partial_{x_k}A\|_\infty(t-s)^{\frac{1}{2}}}{2}+da_p\|\sigma\|_\infty\right]\left\|\left(\int_s^t|D^2F_l(v,X_v)|^2dv\right)^{\frac{1}{2}}\right\|_p
\end{aligned}
$$

where $a_p > 0$ is the constant from the Burkholder-Davis-Gundy inequality, so that

$$\begin{aligned}&\|\nabla_x F_l(t, X_t) - \nabla_x F_l(s, X_s)\|_p \\ &\qquad \leq c_1 \left\| \int_s^t |\nabla_x F_l(v, X_v)| dv \right\|_p + c_2 \left\| \left(\int_s^t |D^2 F_l(v, X_v)|^2 dv \right)^{\frac{1}{2}} \right\|_p \end{aligned} \tag{12}$$

with

$$c_1 := \sum_{k=1}^d \|\partial_{x_k} b\|_\infty \quad \text{and} \quad c_2 := \frac{\sqrt{T}}{2} \sum_{k=1}^d \|\partial_{x_k} A\|_\infty + d a_p \|\sigma\|_\infty.$$

Using this relation for $s = r_{l-1}$ and applying Gronwall's lemma implies

$$\begin{aligned}&\|\nabla_x F_l(t, X_t)\|_p \\ &\qquad \leq e^{c_1 T} \left[\|\nabla_x F_l(r_{l-1}, X_{r_{l-1}})\|_p + c_2 \left\| \left(\int_{r_{l-1}}^t |D^2 F_l(r, X_r)|^2 dr \right)^{\frac{1}{2}} \right\|_p \right].\end{aligned}$$

Now we return to (12) and get that

$$\begin{aligned}
& \|\nabla_x F_l(t, X_t) - \nabla_x F_l(s, X_s)\|_p \\
\leq \;& c_1 \int_s^t \|\nabla_x F_l(r, X_r)\|_p dr + c_2 \left\| \left(\int_s^t |D^2 F_l(r, X_r)|^2 dr \right)^{\frac{1}{2}} \right\|_p \\
\leq \;& c_1 e^{c_1 T} \int_s^t \left[\|\nabla_x F_l(r_{l-1}, X_{r_{l-1}})\|_p + c_2 \left\| \left(\int_{r_{l-1}}^r |D^2 F_l(v, X_v)|^2 dv \right)^{\frac{1}{2}} \right\|_p \right] dr \\
& + c_2 \left\| \left(\int_s^t |D^2 F_l(r, X_r)|^2 dr \right)^{\frac{1}{2}} \right\|_p \\
\leq \;& c_1 e^{c_1 T} (t - s) \|\nabla_x F_l(r_{l-1}, X_{r_{l-1}})\|_p \\
& + c_1 c_2 e^{c_1 T} \int_s^t \left\| \left(\int_{r_{l-1}}^s |D^2 F_l(v, X_v)|^2 dv \right)^{\frac{1}{2}} \right\|_p dr \\
& + c_1 c_2 e^{c_1 T} \int_s^t \left\| \left(\int_s^r |D^2 F_l(v, X_v)|^2 dv \right)^{\frac{1}{2}} \right\|_p dr \\
& + c_2 \left\| \left(\int_s^t |D^2 F_l(r, X_r)|^2 dr \right)^{\frac{1}{2}} \right\|_p \\
\leq \;& c_1 e^{c_1 T} (t - s) \|\nabla_x F_l(r_{l-1}, X_{r_{l-1}})\|_p
\end{aligned}$$

$$+(t-s)c_1c_2e^{c_1T}\left\|\left(\int_{r_{l-1}}^{s}|D^2F_l(v,X_v)|^2dv\right)^{\frac{1}{2}}\right\|_p$$

$$+[c_1c_2e^{c_1T}(t-s)+c_2]\left\|\left(\int_{s}^{t}|D^2F_l(v,X_v)|^2dv\right)^{\frac{1}{2}}\right\|_p.$$

□

For $r\in[r_{l-1},r_l)$ we consider

$$\delta v_l(\overline{x}_{l-1};r,x):=v_l(\overline{x}_{l-1};r,x)-\nabla_xF_l(\overline{x}_{l-1};r,x) \tag{13}$$

and get that, a.s.,

$$\begin{aligned}u_l(\overline{x}_{l-1};r_{l-1},x_{l-1})-F_l(\overline{x}_{l-1};r_{l-1},x_{l-1})=&\\ &\int_{r_{l-1}}^{r_l}\overline{f}(\overline{x}_{l-1};r,X_r^{r_{l-1},x_{l-1}})dr\\ &-\int_{r_{l-1}}^{r_l}\delta v_l(\overline{x}_{l-1};r,X_r^{r_{l-1},x_{l-1}})\sigma(r,X_r^{r_{l-1},x_{l-1}})dW_r^{r_{l-1}}\end{aligned}$$

with

$$\overline{f}(\overline{x}_{l-1};r,x):=f(r,x,u_l(\overline{x}_{l-1};r,x),v_l(\overline{x}_{l-1};r,x)\sigma(r,x)).$$

Letting

$$\lambda^r(\overline{x}_{l-1};s,x):=\int_{\mathbb{R}^d}\overline{f}(\overline{x}_{l-1};r,\xi)\nabla_x\Gamma(s,x;r,\xi)d\xi$$

and applying a stochastic Fubini argument, it follows that

$$\begin{aligned}&\delta v_l(\overline{x}_{l-1};s,X_s^{r_{l-1},x_{l-1}})\sigma(s,X_s^{r_{l-1},x_{l-1}})\\ &\qquad=\int_s^{r_l}\lambda^r(\overline{x}_{l-1};s,X_s^{r_{l-1},x_{l-1}})dr\ \sigma(s,X_s^{r_{l-1},x_{l-1}})\quad a.s.\end{aligned}$$

for $s\in[r_{l-1},r_l)\setminus\mathcal{N}_l(\overline{x}_{l-1})$, where $\mathcal{N}_l(\overline{x}_{l-1})$ is a Borel set of measure zero. Hence for $s\in[r_{l-1},r_l)\setminus\mathcal{N}_l(\overline{x}_{l-1})$ we get by (2) and Proposition 1.1 that

$$\begin{aligned}&\|\delta v_l(\overline{x}_{l-1};s,X_s^{r_{l-1},x_{l-1}})\sigma(s,X_s^{r_{l-1},x_{l-1}})\|_p\\ \leq\ &\int_s^{r_l}\|\lambda^r(\overline{x}_{l-1};s,X_s^{r_{l-1},x_{l-1}})\sigma(s,X_s^{r_{l-1},x_{l-1}})\|_pdr\\ \leq\ &\|\sigma\|_\infty\,\kappa_{p'}\int_s^{r_l}\frac{\|f(\overline{x}_{l-1};r,X_r^{r_{l-1},x_{l-1}})\|_p}{\sqrt{r-s}}dr\\ \leq\ &\|\sigma\|_\infty\kappa_{p'}\int_s^{r_l}\Bigg[\frac{K_f+L_f\big[\|X_r^{r_{l-1},x_{l-1}}\|_p+\|u_l(\overline{x}_{l-1};r,X_r^{r_{l-1},x_{l-1}})\|_p\big]}{\sqrt{r-s}}\\ &\qquad+\frac{L_f\|v_l(\overline{x}_{l-1};r,X_r^{r_{l-1},x_{l-1}})\sigma(r,X_r^{r_{l-1},x_{l-1}})\|_p}{\sqrt{r-s}}\Bigg]dr\end{aligned}$$

$$
\begin{aligned}
\leq \quad & \|\sigma\|_\infty \kappa_{p'} \int_s^{r_l} \frac{1}{\sqrt{r-s}} \Bigg(K_f + L_f \Bigg[\|X_r^{r_{l-1},x_{l-1}}\|_p + \\
& \alpha_l \left(1 + \frac{\|\sigma\|_\infty}{\sqrt{r_l - r}} \right) \|1 + |x_1|^{q_{l,1}} + \cdots + |x_{l-1}|^{q_{l,l-1}} + |X_r^{r_{l-1},x_{l-1}}|^{q_{l,l}} \|_p \Bigg] \Bigg) dr.
\end{aligned}
$$

By continuity of both sides in s one can estimate the first term by the last term in the above display for all $s \in [r_{l-1}, r_l)$. Using the stochastic flow we obtain the inequality

$$
\begin{aligned}
& \|Z_s - \nabla_x F_l(\overline{X}_{l-1}; s, X_s) \sigma(s, X_s)\|_p \\
\leq \quad & \|\sigma\|_\infty \kappa_{p'} \int_s^{r_l} \frac{1}{\sqrt{r-s}} \Bigg(K_f + L_f \Bigg[\|X_r\|_p + \\
& \alpha_l \left(1 + \frac{\|\sigma\|_\infty}{\sqrt{r_l - r}} \right) \left\| 1 + |X_{r_1}|^{q_{l,1}} + \cdots + |X_{r_{l-1}}|^{q_{l,l-1}} + |X_r|^{q_{l,l}} \right\|_p \Bigg] \Bigg) dr \\
\leq \quad & c_0 < \infty
\end{aligned}
$$

where $c_0 > 0$ does not depend on s. The assertion $(C2_l)$ follows from this and Lemma 3.1 applied to $s = r_{l-1}$ because

$$
\begin{aligned}
\|Z_r\|_p \quad \leq \quad & \|Z_r - \nabla_x F_l(\overline{X}_{l-1}; r, X_r)\sigma(r, X_r)\|_p + \|\sigma\|_\infty \|\nabla_x F_l(\overline{X}_{l-1}; r, X_r)\|_p \\
\leq \quad & c_0 + \|\sigma\|_\infty (1 + c_{(3.1)} T) \|\nabla_x F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}})\|_p \\
& + \|\sigma\|_\infty c_{(3.1)} \left\| \left(\int_{r_{l-1}}^t |D^2 F_l(\overline{X}_{l-1}; v, X_v)|^2 dv \right)^{\frac{1}{2}} \right\|_p .
\end{aligned}
$$

$\underline{(C4_l) \Longrightarrow (C1_l)}$ To make our assumption $(C4_l)$ more transparent, the constant $\underline{c_4 > 0 \text{ of this}}$ condition is denoted by $c_{B_{p,\infty}^\Theta}$ in the following. Using (13) and letting $r_{l-1} \leq r < r_l$, by condition $(A_{b,\sigma})$ we get that

$$
\begin{aligned}
& \|Z_t^{r_{l-1},\overline{x}_{l-1}} - Z_s^{r_{l-1},\overline{x}_{l-1}}\|_p \\
\leq \quad & \|Z_t^{r_{l-1},\overline{x}_{l-1}} \sigma(t, X_t^{r_{l-1},x_{l-1}})^{-1} - Z_s^{r_{l-1},\overline{x}_{l-1}} \sigma(s, X_s^{r_{l-1},x_{l-1}})^{-1}\|_p \|\sigma\|_\infty \\
& + \|Z_s^{r_{l-1},\overline{x}_{l-1}} \sigma(s, X_s^{r_{l-1},x_{l-1}})^{-1} (\sigma(t, X_t^{r_{l-1},x_{l-1}}) - \sigma(s, X_s^{r_{l-1},x_{l-1}}))\|_p \\
\leq \quad & \|\nabla_x F_l(\overline{x}_{l-1}; t, X_t^{r_{l-1},x_{l-1}}) - \nabla_x F_l(\overline{x}_{l-1}; s, X_s^{r_{l-1},x_{l-1}})\|_p \|\sigma\|_\infty \\
& + \|\delta v_l(\overline{x}_{l-1}; t, X_t^{r_{l-1},x_{l-1}}) - \delta v_l(\overline{x}_{l-1}; s, X_s^{r_{l-1},x_{l-1}})\|_p \|\sigma\|_\infty \\
& + L_\sigma \|\sigma^{-1}\|_\infty \|Z_s^{r_{l-1},\overline{x}_{l-1}}\|_p \times \\
& (\|\mathbb{E}(\left|X_t^{r_{l-1},x_{l-1}} - X_s^{r_{l-1},x_{l-1}}\right|^p |\mathcal{F}_s^{r_{l-1}})\|_\infty^{\frac{1}{p}} + |t-s|^{\frac{1}{2}}) \\
\leq \quad & c_{\sigma,b,p,T} \left[D_1(\overline{x}_{l-1}) + D_2(\overline{x}_{l-1}) + D_3(\overline{x}_{l-1}) \right]
\end{aligned}
$$

with

$$
\begin{aligned}
D_1(\overline{x}_{l-1}) \quad & := \quad \|\nabla_x F_l(\overline{x}_{l-1}; t, X_t^{r_{l-1},x_{l-1}}) - \nabla_x F_l(\overline{x}_{l-1}; s, X_s^{r_{l-1},x_{l-1}})\|_p, \\
D_2(\overline{x}_{l-1}) \quad & := \quad \|\delta v_l(\overline{x}_{l-1}; t, X_t^{r_{l-1},x_{l-1}}) - \delta v_l(\overline{x}_{l-1}; s, X_s^{r_{l-1},x_{l-1}})\|_p, \\
D_3(\overline{x}_{l-1}) \quad & := \quad (t-s)^{\frac{1}{2}} \|Z_s^{r_{l-1},\overline{x}_{l-1}}\|_p.
\end{aligned}
$$

Now we show that each $\|D_i(\overline{X}_{l-1})\|_p$, $i = 1, 2, 3$, is bounded by a constant times $\left(\int_s^t (r_l - r)^{\theta_l - 2} dr\right)^{\frac{1}{2}}$ which implies $(C1_l)$.

The term $D_1(\overline{X}_{l-1})$: Here we use Lemma 3.1 to get

$$\begin{aligned}
\|D_1(\overline{X}_{l-1})\|_p &= \|\nabla_x F_l(\overline{X}_{l-1}; t, X_t) - \nabla_x F_l(\overline{X}_{l-1}; s, X_s)\|_p \\
&\le c_{(3.1)}(t-s)\|\nabla_x F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}})\|_p \\
&\quad + c_{(3.1)}(t-s)\left\|\left(\int_{r_{l-1}}^s |D^2 F_l(\overline{X}_{l-1}; v, X_v)|^2 dv\right)^{\frac{1}{2}}\right\|_p \\
&\quad + c_{(3.1)}\left\|\left(\int_s^t |D^2 F_l(\overline{X}_{l-1}; v, X_v)|^2 dv\right)^{\frac{1}{2}}\right\|_p \\
&\le c_{(3.1)}(t-s)\|\nabla_x F_l(\overline{X}_{l-1}; r_{l-1}, X_{r_{l-1}})\|_p \\
&\quad + c_{(3.1)}(t-s)\left(\int_{r_{l-1}}^s \|D^2 F_l(\overline{X}_{l-1}; v, X_v)\|_p^2 dv\right)^{\frac{1}{2}} \\
&\quad + c_{(3.1)}\left(\int_s^t \|D^2 F_l(\overline{X}_{l-1}; v, X_v)\|_p^2 dv\right)^{\frac{1}{2}} \\
&\le c_{(3.1)}(t-s)\kappa_{p'} c_{B^\Theta_{p,\infty}}(r_l - r_{l-1})^{\frac{\theta_l - 1}{2}} \\
&\quad + c_{(3.1)}(t-s)\left(\int_{r_{l-1}}^s \kappa_{p'}^2 c^2_{B^\Theta_{p,\infty}}(r_l - v)^{\theta_l - 2} dv\right)^{\frac{1}{2}} \\
&\quad + c_{(3.1)}\left(\int_s^t \kappa_{p'}^2 c^2_{B^\Theta_{p,\infty}}(r_l - v)^{\theta_l - 2} dv\right)^{\frac{1}{2}}
\end{aligned}$$

where we have used (3). Finally we apply

$$\begin{aligned}
(t-s)\left(\int_{r_{l-1}}^s (r_l - v)^{\theta_l - 2} dv\right)^{\frac{1}{2}} &\le (t-s)\sqrt{s - r_{l-1}}(r_l - s)^{\frac{\theta_l - 2}{2}} \\
&\le \sqrt{t-s}\sqrt{s - r_{l-1}}\left(\int_s^t (r_l - v)^{\theta_l - 2} dv\right)^{\frac{1}{2}}.
\end{aligned}$$

The term $D_2(\overline{x}_{l-1})$ and a linearization: First we follow the approach of [14] done for the one-step scheme, that shows that the difference process $((v_l - \nabla_x F_l)(\overline{X}_{l-1}; r, X_r))_{r \in [r_{l-1}, r_l)}$ solves the linear BSDE with the generator f^{lin} defined below. We fix $x_1, ..., x_{l-1} \in \mathbb{R}^d$ and define $f^{\text{lin}} : [r_{l-1}, r_l) \times \mathbb{R}^d \times \mathbb{R}^{1\times d} \times \mathbb{R}^{d\times d} \to \mathbb{R}^{1\times d}$ by

$$f^{\text{lin}}(\overline{x}_{l-1}; r, x, U, V) := A_l^0(\overline{x}_{l-1}; r, x) + U B_l^0(\overline{x}_{l-1}; r, x) + \sum_{j=1}^d V_j C_l^{j,0}(\overline{x}_{l-1}; r, x),$$

where V_j is the j-th row of V, with

$$
\begin{aligned}
A_l^0(\overline{x}_{l-1}; r, x) &:= \nabla_x f\Big(r, x, u_l(\overline{x}_{l-1}; r, x), v_l(\overline{x}_{l-1}; r, x)\sigma(r, x)\Big) \\
&\quad + \frac{\partial f}{\partial y}\Big(r, x, u_l(\overline{x}_{l-1}; r, x), v_l(\overline{x}_{l-1}; r, x)\sigma(r, x)\Big)\nabla_x F_l(\overline{x}_{l-1}; r, x) \\
&\qquad + \sum_{j=1}^{d} \frac{\partial f}{\partial z_j}\Big(r, x, u_l(\overline{x}_{l-1}; r, x), v_l(\overline{x}_{l-1}; r, x)\sigma(r, x)\Big)\times \\
&\qquad\qquad \times \nabla_x \left(\sum_{k=1}^{d} \frac{\partial F_l}{\partial x_k}(\overline{x}_{l-1}; r, x)\sigma_{kj}(r, x)\right), \\
B_l^0(\overline{x}_{l-1}; r, x) &:= \frac{\partial f}{\partial y}(r, x, u_l(\overline{x}_{l-1}; r, x), v_l(\overline{x}_{l-1}; r, x)\sigma(r, x))I_{\mathbb{R}^d} \\
&\quad + \nabla_x b(r, x) + \sum_{j=1}^{d} \frac{\partial f}{\partial z_j}(r, x, u_l(\overline{x}_{l-1}; r, x), v_l(\overline{x}_{l-1}; r, x)\sigma(r, x))\nabla_x \sigma_j(r, x)
\end{aligned}
$$

and

$$
\begin{aligned}
&C_l^{j,0}(\overline{x}_{l-1}; r, x) \\
&\qquad := \frac{\partial f}{\partial z_j}(r, x, u_l(\overline{x}_{l-1}; r, x), v_l(\overline{x}_{l-1}; r, x)\sigma(r, x))I_{\mathbb{R}^d} + \nabla_x \sigma_j(r, x),
\end{aligned}
$$

with $\sigma_j = (\sigma_{kj})_{k=1}^d \in \mathbb{R}^d$, δv_l defined as in (13), and

$$
\delta u_l(\overline{x}_{l-1}; r, x) := u_l(\overline{x}_{l-1}; r, x) - F_l(\overline{x}_{l-1}; r, x).
$$

This implies

$$
|f^{\mathrm{lin}}(\overline{x}_{l-1}; r, x, u, v)| \leq |A_l^0(\overline{x}_{l-1}; r, x)| + c_{(14)}[|u| + |v|]. \tag{14}
$$

To associate a BSDE to the driver f^{lin}, we first check that

$$
\int_{r_{l-1}}^{r_l} \|A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1}, x_{l-1}})\|_p \, dr < \infty. \tag{15}
$$

For this purpose we let

$$
\psi_l(\overline{x}_{l-1}; r) := 1 + \|\nabla_x F_l(\overline{x}_{l-1}; r, X_r^{r_{l-1}, x_{l-1}})\|_p + \|D^2 F_l(\overline{x}_{l-1}; r, X_r^{r_{l-1}, x_{l-1}})\|_p,
$$

which implies that

$$
\|A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1}, x_{l-1}})\|_p \leq c_{(16)} \psi_l(\overline{x}_{l-1}; r). \tag{16}
$$

In view of (2) and (3) we have that

$$\psi_l(\overline{x}_{l-1};r) \leq 1 + (1+\sqrt{r_l - r})\frac{\kappa_{p'}}{r_l - r}\times$$
$$\left\| F_l(\overline{x}_{l-1}; r_l, X_{r_l}^{r_{l-1},x_{l-1}}) - F_l(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}})\right\|_p. \quad (17)$$

To obtain the integrability of the upper bound on $\psi_l(\overline{x}_{l-1};r)$ (and thus that of $\|A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}})\|_p$), we show that the assumption on global fractional smoothness implies a local fractional smoothness. Indeed, our global assumption reads as

$$\left\|F_l(\overline{X}_{l-1}; r_l, X_{r_l}) - F_l(\overline{X}_{l-1}; s, X_s)\right\|_p \leq c_{B^{\Theta}_{p,\infty}} (r_l - s)^{\frac{\theta_l}{2}} \quad (18)$$

for $r_{l-1} \leq s < r_l$. For any $0 < \delta < 1$ this implies that

$$\int_{r_{l-1}}^{r_l} (r_l - s)^{-\frac{p\theta_l}{2} - \delta} \left\|F_l(\overline{X}_{l-1}; r_l, X_{r_l}) - F_l(\overline{X}_{l-1}; s, X_s)\right\|_p^p ds < \infty.$$

Using the transition density of X and Fubini's theorem implies the existence of a Borel set $E_l \subseteq (\mathbb{R}^d)^{l-1}$ such that E_l^c has Lebesgue measure zero and

$$\int_{r_{l-1}}^{r_l} (r_l - s)^{-\frac{p\theta_l}{2} - \delta} \left\|F_l(\overline{x}_{l-1}; r_l, X_{r_l}^{r_{l-1},x_{l-1}}) - F_l(\overline{x}_{l-1}; s, X_s^{r_{l-1},x_{l-1}})\right\|_p^p ds < \infty$$

for all $(x_1, ..., x_{l-1}) \in E_l$. For those $(x_1, ..., x_{l-1}) \in E_l$ we may deduce (using (5)) for $s \in ((r_{l-1} + r_l)/2, r_l)$ and $a_l := s - (r_l - s)$ that

$$\begin{aligned}
& \left\|F_l(\overline{x}_{l-1}; r_l, X_{r_l}^{r_{l-1},x_{l-1}}) - F_l(\overline{x}_{l-1}; s, X_s^{r_{l-1},x_{l-1}})\right\|_p^p \\
\leq\ & 2^p (s - a_l)^{-1} (r_l - a_l)^{\delta + \frac{p\theta_l}{2}} \\
& \int_{a_l}^{s} (r_l - r)^{-\frac{p\theta_l}{2} - \delta} \left\|F_l(\overline{x}_{l-1}; r_l, X_{r_l}^{r_{l-1},x_{l-1}}) - F_l(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}})\right\|_p^p dr \\
\leq\ & 2^{p+\delta+\frac{p\theta_l}{2}} (r_l - s)^{\delta + \frac{p\theta_l}{2} - 1} \\
& \int_{r_{l-1}}^{r_l} (r_l - r)^{-\frac{p\theta_l}{2} - \delta} \left\|F_l(\overline{x}_{l-1}; r_l, X_{r_l}^{r_{l-1},x_{l-1}}) - F_l(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}})\right\|_p^p dr.
\end{aligned}$$

Taking $0 < \delta < 1$ such that $\delta + \frac{p\theta_l}{2} - 1 > 0$ we obtain a local fractional smoothness for all $(x_1, ..., x_{l-1}) \in E_l$. Then for $\overline{x}_{l-1} \in E_l$ the inequality (15) is satisfied. Thus, because of [14, Theorem 2.1] the process $(\delta v_l(\overline{x}_{l-1}; s, X_s^{r_{l-1},x_{l-1}}))_{s\in[r_{l-1},r_l)}$ solves the U-component of the BSDE

$$\begin{aligned}
U_s^{r_{l-1},\overline{x}_{l-1}} &= \int_s^{r_l} f^{\mathrm{lin}}(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}}, U_r^{r_{l-1},\overline{x}_{l-1}}, V_r^{r_{l-1},\overline{x}_{l-1}}) dr \\
& \quad - \left(\int_s^{r_l} (V_r^{r_{l-1},\overline{x}_{l-1}})^* dW_r^{r_{l-1}}\right)^*
\end{aligned}$$

for all $\overline{x}_{l-1} \in E_l$ (according to (14), (15) and [6, Theorem 4.2] this BSDE has a unique L_p-solution).

Upper bound for $\|D_2(\overline{X}_{l-1})\|_p$: Applying Lemma A.3 to $h = f^{\mathrm{lin}}$ (the function κ from Lemma A.3(iii) is obtained by Proposition B.1 and (15) is used) it follows that

$$
\begin{aligned}
& \|U_s^{r_{l-1},\overline{x}_{l-1}}\|_p \\
\leq\ & c_{(A.3)} \left\| \int_s^{r_l} |A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}})| dr \right\|_p \\
\leq\ & c_{(A.3)} c_{(16)} \int_s^{r_l} \psi_l(\overline{x}_{l-1}; r) dr \\
\leq\ & c_{(A.3)} c_{(16)} \Bigg[[r_l - s] + \kappa_{p'} \int_s^{r_l} (1 + \sqrt{r_l - r}) \\
& \qquad \times \frac{\left\| F_l(\overline{x}_{l-1}; r_l, X_{r_l}^{r_{l-1},x_{l-1}}) - F_l(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}}) \right\|_p}{r_l - r} dr \Bigg] \\
=:\ & \varphi_l(\overline{x}_{l-1}; s),
\end{aligned}
$$

that means

$$
\|U_s^{r_{l-1},\overline{x}_{l-1}}\|_p \leq \varphi_l(\overline{x}_{l-1}; s) \tag{19}
$$

with

$$
\left\| \varphi_l(\overline{X}_{l-1}; s) \right\|_p \leq c_{(A.3)} c_{(16)} \left[[r_l - s] + \kappa_{p'} (1 + \sqrt{T}) c_{B^{\Theta}_{p,\infty}} \int_s^{r_l} (r_l - r)^{\frac{\theta_l}{2} - 1} dr \right]
$$

or

$$
\left\| \varphi_l(\overline{X}_{l-1}; s) \right\|_p \leq c_{(20)} \left[[r_l - s] + c_{B^{\Theta}_{p,\infty}} \int_s^{r_l} (r_l - r)^{\frac{\theta_l}{2} - 1} dr \right]. \tag{20}
$$

Exploiting again Lemma A.3 also gives that

$$
\begin{aligned}
\|V_s^{r_{l-1},\overline{x}_{r-1}}\|_p & \leq c_{(A.3)} \int_s^{r_l} \frac{\|A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}})\|_p}{\sqrt{r-s}} dr \\
& \leq c_{(A.3)} c_{(16)} \int_s^{r_l} \frac{\psi_l(\overline{x}_{l-1}; r)}{\sqrt{r-s}} dr
\end{aligned}
$$

for $s \in [r_{l-1}, r_l) \setminus \mathcal{N}_l(\overline{x}_{l-1})$, where $\mathcal{N}_l(\overline{x}_{l-1})$ has Lebesgue measure zero. Hence

$$
\begin{aligned}
& \|U_s^{r_{l-1},\overline{x}_{l-1}} - U_t^{r_{l-1},\overline{x}_{l-1}}\|_p \\
=\ & \Bigg\| \int_s^t f^{\mathrm{lin}}(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}}, U_r^{r_{l-1},\overline{x}_{l-1}}, V_r^{r_{l-1},\overline{x}_{l-1}}) dr \\
& \qquad\qquad\qquad - \int_s^t V_r^{r_{l-1},\overline{x}_{l-1}} dW_r^{r_{l-1}} \Bigg\|_p \\
\leq\ & \left\| \int_s^t |A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1},x_{l-1}})| dr \right\|_p \\
& + c_{(14)} \left\| \int_s^t [|U_r^{r_{l-1},\overline{x}_{l-1}}| + |V_r^{r_{l-1},\overline{x}_{l-1}}|] dr \right\|_p \\
& + a_p \left\| \left(\int_s^t |V_r^{r_{l-1},\overline{x}_{l-1}}|^2 dr \right)^{\frac{1}{2}} \right\|_p
\end{aligned}
$$

$$\le \left\| \int_s^t |A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1}, x_{l-1}})| dr \right\|_p + c_{(14)} \int_s^t \|U_r^{r_{l-1}, \overline{x}_{l-1}}\|_p dr$$

$$+ [c_{(14)}\sqrt{t-s} + a_p] \left(\int_s^t \|V_r^{r_{l-1}, \overline{x}_{l-1}}\|_p^2 dr \right)^{\frac{1}{2}}$$

$$\le \left\| \int_s^t |A_l^0(\overline{x}_{l-1}; r, X_r^{r_{l-1}, x_{l-1}})| dr \right\|_p + c_{(14)} \int_s^t \varphi_l(\overline{x}_{l-1}; r) dr$$

$$+ [c_{(14)}\sqrt{t-s} + a_p] c_{(A.3)} c_{(16)} \left(\int_s^t \left| \int_r^{r_l} \frac{\psi_l(\overline{x}_{l-1}; w)}{\sqrt{w-r}} dw \right|^2 dr \right)^{\frac{1}{2}}.$$

Because $\mathbb{P}((X_{r_1}, ..., X_{r_{l-1}}) \in E_l) = 1$ we can use the stochastic flow property and can bound $\|D_2(\overline{X}_{l-1})\|_p$ from above by the L_p-norms of the following three expressions: Taking the L_p-norm of the last term gives

$$\left\| \left(\int_s^t \left| \int_r^{r_l} \frac{\psi_l(\overline{X}_{l-1}; w)}{\sqrt{w-r}} dw \right|^2 dr \right)^{\frac{1}{2}} \right\|_p$$

$$\le \left(\int_s^t \left| \int_r^{r_l} \frac{1 + \|\nabla_x F_l(\overline{X}_{l-1}; w, X_w)\|_p + \|D^2 F_l(\overline{X}_{l-1}; w, X_w)\|_p}{\sqrt{w-r}} dw \right|^2 dr \right)^{\frac{1}{2}}$$

$$\le \left(\int_s^t \left| \int_r^{r_l} \frac{dw}{\sqrt{w-r}} \right|^2 dr \right)^{\frac{1}{2}}$$

$$+ \kappa_{p'} c_{B_{p,\infty}^\Theta} \left(\int_s^t \left| \int_r^{r_l} \frac{(r_l - w)^{\frac{\theta_l - 1}{2}} + (r_l - w)^{\frac{\theta_l - 2}{2}}}{\sqrt{w-r}} dw \right|^2 dr \right)^{\frac{1}{2}}$$

$$\le \left(\int_s^t \left| \int_r^{r_l} \frac{dw}{\sqrt{w-r}} \right|^2 dr \right)^{\frac{1}{2}}$$

$$+ \kappa_{p'} c_{B_{p,\infty}^\Theta} (1 + \sqrt{T}) \left(\int_s^t \left| \int_r^{r_l} \frac{(r_l - w)^{\frac{\theta_l - 2}{2}}}{\sqrt{w-r}} dw \right|^2 dr \right)^{\frac{1}{2}}$$

$$\le 2\sqrt{T}\sqrt{t-s} + \kappa_{p'} c_{B_{p,\infty}^\Theta} (1 + \sqrt{T}) \gamma_l \left(\int_s^t (r_l - r)^{\theta_l - 1} dr \right)^{\frac{1}{2}}$$

with $\gamma_l := \int_0^1 \frac{(1-t)^{\frac{\theta_l}{2} - 1}}{\sqrt{t}} dt$. For the next to the last term we obtain

$$\left\| \int_s^t \varphi_l(\overline{X}_{l-1}; r) dr \right\|_p \le c_{(20)} \int_s^t \left[(r_l - r) + c_{B_{p,\infty}^\Theta} \int_r^{r_l} (r_l - w)^{\frac{\theta_l}{2} - 1} dw \right] dr$$

$$\le c_{(20)} \left[T + c_{B_{p,\infty}^\Theta} \frac{2}{\theta_l} T^{\frac{\theta_l}{2}} \right] (t-s).$$

Finally, we get by (17) and (18) that

$$
\begin{aligned}
& \left\| \int_s^t |A_l^0(\overline{X}_{l-1}; r, X_r)| dr \right\|_p \\
\leq \;& c_{(16)} \int_s^t \|\psi_l(\overline{X}_{l-1}; r)\|_p dr \\
\leq \;& c_{(16)} \left[(t-s) + \sqrt{T}(1+\sqrt{T}) \kappa_{p'} c_{B^{\Theta}_{p,\infty}} \left(\int_s^t (r_l - r)^{\theta_l - 2} dr \right)^{\frac{1}{2}} \right].
\end{aligned}
$$

<u>The term $D_3(\overline{X}_{l-1})$:</u> Let $r_{l-1} \leq s < t < r_l$ and recall

$$
Z_t^{r_{l-1}, \overline{x}_{l-1}} = v_l(\overline{x}_{l-1}; t, X_t^{r_{l-1}, x_{l-1}}) \sigma(t, X_t^{r_{l-1}, x_{l-1}}).
$$

From inequality (19) we obtain

$$
\begin{aligned}
& (t-s)^{\frac{1}{2}} \|Z_s^{r_{l-1}, \overline{X}_{l-1}}\|_p \\
\leq \;& (t-s)^{\frac{1}{2}} \|\sigma\|_\infty \|v_l(\overline{X}_{l-1}; s, X_s^{r_{l-1}, X_{r_{l-1}}})\|_p \\
\leq \;& (t-s)^{\frac{1}{2}} \|\sigma\|_\infty \left[\|\nabla_x F_l(\overline{X}_{l-1}; s, X_s^{r_{l-1}, X_{r_{l-1}}})\|_p + \|U_s^{r_{l-1}, \overline{X}_{l-1}}\|_p \right] \\
\leq \;& (t-s)^{\frac{1}{2}} \|\sigma\|_\infty (\kappa_{p'} c_{B^{\Theta}_{p,\infty}} (r_l - s)^{\frac{\theta_l - 1}{2}} + \|\varphi_l(\overline{X}_{l-1}, s)\|_p) \\
\leq \;& (t-s)^{\frac{1}{2}} \|\sigma\|_\infty \\
& \left(\kappa_{p'} c_{B^{\Theta}_{p,\infty}} (r_l - s)^{\frac{\theta_l - 1}{2}} + c_{(20)} \left[[r_l - s] + c_{B^{\Theta}_{p,\infty}} \int_s^{r_l} (r_l - r)^{\frac{\theta_l}{2} - 1} dr \right] \right) \\
\leq \;& c(t-s)^{\frac{1}{2}} [1 + (r_l - s)^{\frac{\theta_l - 1}{2}}] \\
\leq \;& c \left[(t-s)^{\frac{1}{2}} + \left(\int_s^t (r_l - r)^{\theta_l - 1} dr \right)^{\frac{1}{2}} \right].
\end{aligned}
$$

<u>$(C3_l) \Longrightarrow (C6_l)$</u> Let

$$
t_k^{n,\theta_l} := r_{l-1} + (r_l - r_{l-1}) \left(1 - \left(1 - \frac{k}{n} \right)^{\frac{1}{\theta_l}} \right) \quad \text{for} \quad k = 0, ..., n
$$

and $S^n_{t_k^{n,\theta_l}} := Y_{t_k^{n,\theta_l}}$. One obtains for $t \in (t_{k-1}^{n,\theta_l}, t_k^{n,\theta_l}) \subseteq [r_{l-1}, r_l]$ and an appropriate $\eta \in (0,1)$, that

$$
\begin{aligned}
& \|S_t^n - Y_t\|_p \\
= \;& \|(1-\eta) Y_{t_{k-1}^{n,\theta_l}} + \eta Y_{t_k^{n,\theta_l}} - Y_t\|_p \\
\leq \;& (1-\eta) \|Y_{t_{k-1}^{n,\theta_l}} - Y_t\|_p + \eta \|Y_{t_k^{n,\theta_l}} - Y_t\|_p \\
\leq \;& (1-\eta) c_3 \left(\int_{t_{k-1}^{n,\theta_l}}^t (r_l - r)^{\theta_l - 1} dr \right)^{\frac{1}{2}} + \eta c_3 \left(\int_t^{t_k^{n,\theta_l}} (r_l - r)^{\theta_l - 1} dr \right)^{\frac{1}{2}}
\end{aligned}
$$

$$\begin{aligned} &\le c_3\left(\frac{1}{\theta_l}[(r_l-t^{n,\theta_l}_{k-1})^{\theta_l}-(r_l-t^{n,\theta_l}_{k})^{\theta_l}]\right)^{\frac{1}{2}}\\ &= c_3\frac{(r_l-r_{l-1})^{\frac{\theta_l}{2}}}{\sqrt{\theta_l}}\frac{1}{\sqrt{n}}. \end{aligned}$$

$\underline{(C6_l) \Longrightarrow (C4_l)}$ We consider

$$\begin{aligned} &\left\|Y_{\frac{r_l+t^{n,\theta_l}_{n-1}}{2}}-S^n_{\frac{r_l+t^{n,\theta_l}_{n-1}}{2}}\right\|_p\\ =&\left\|Y_{\frac{r_l+t^{n,\theta_l}_{n-1}}{2}}-\frac{1}{2}\left[S^n_{r_l}+S^n_{t^{n,\theta_l}_{n-1}}\right]\right\|_p\\ \ge&\left\|Y_{\frac{r_l+t^{n,\theta_l}_{n-1}}{2}}-\frac{1}{2}\left[Y_{r_l}+S^n_{t^{n,\theta_l}_{n-1}}\right]\right\|_p-\frac{1}{2}\|Y_{r_l}-S^n_{r_l}\|_p \end{aligned}$$

so that

$$\left\|Y_{r_l}-2Y_{\frac{r_l+t^{n,\theta_l}_{n-1}}{2}}+S^n_{t^{n,\theta_l}_{n-1}}\right\|_p\le\frac{3c_6}{\sqrt{n}}.$$

But this means that

$$\left\|Y_{r_l}-\mathbb{E}\left(Y_{r_l}|\mathcal{F}_{\frac{r_l+t^{n,\theta_l}_{n-1}}{2}}\right)\right\|_p\le\frac{6c_6}{\sqrt{n}}.$$

Because

$$r_l-\frac{r_l+t^{n,\theta_l}_{n-1}}{2}=\frac{1}{2}(r_l-r_{l-1})n^{-\frac{1}{\theta_l}}$$

we get that

$$\|Y_{r_l}-\mathbb{E}\left(Y_{r_l}|\mathcal{F}_t\right)\|_p\le 6c_6\left(\frac{r_l-r_{l-1}}{2}\right)^{-\frac{\theta_l}{2}}(r_l-t)^{\frac{\theta_l}{2}}\quad\text{for}\quad t=\frac{r_l+t^{n,\theta_l}_{n-1}}{2}.$$

Using (5) proves our assertion for $r_{l-1}+\frac{r_l-r_{l-1}}{2}\le t<r_l$. For the remaining $r_{l-1}\le t<r_{l-1}+\frac{r_l-r_{l-1}}{2}$ we can simply use $\|Y_{r_l}-Y_{r_{l-1}}\|_p<\infty$.

$\underline{(C7_l) \Longrightarrow (C4_l)}$ Let $t\in[r_{l-1},r_l)$. We use $(C7_l)$ for $n=1$ so that Y_t and Y_{r_l} can be covered by *one* ball with any radius bigger than $c_7(r_l-t)^{\frac{\theta_l}{2}}$. Taking the infimum of these radii we get that $\|Y_{r_l}-Y_t\|_p\le 2c_7(r_l-t)^{\frac{\theta_l}{2}}$ which implies that

$$\|Y_{r_l}-\mathbb{E}(Y_{r_l}|\mathcal{F}_t)\|_p\le 4c_7(r_l-t)^{\frac{\theta_l}{2}}.$$

$\underline{(C3_l) \Longrightarrow (C7_l)}$ Fix $t\in[r_{l-1},r_l)$ and $n\ge 1$. Let $N\ge 1$ and choose $k\in\{1,...,N\}$ such that

$$t\in[t^{N,\theta_l}_{k-1},t^{N,\theta_l}_{k})\subseteq[r_{l-1},r_l).$$

For those time-nets we computed in $(C3) \Longrightarrow (C6)$ that

$$\|Y_u - Y_v\|_p \le c_3 \frac{(r_l - r_{l-1})^{\frac{\theta_l}{2}}}{\sqrt{\theta_l}} \frac{1}{\sqrt{N}}$$

for $u, v \in [t_{k-1}^{N,\theta_l}, t_k^{N,\theta_l}] \subseteq [r_{l-1}, r_l]$. Now we choose $N \ge 1$ such that the cardinality of $\left\{ t_k^{N,\theta_l} : k = 0, ..., N \right\} \cap [t_k^{N,\theta_l}, r_l]$ is equal to n, i.e.

$$n = 1 + N \left(\frac{r_l - t_k^{N,\theta_l}}{r_l - r_{l-1}} \right)^{\theta_l}.$$

For $n \ge 2$ this implies that

$$\frac{n}{2} \le n - 1 = \frac{N}{(r_l - r_{l-1})^{\theta_l}} (r_l - t_k^{N,\theta_l})^{\theta_l} \le \frac{N}{(r_l - r_{l-1})^{\theta_l}} (r_l - t)^{\theta_l}$$

and

$$e_n((Y_s)_{s \in [t, r_l]} | L_p) \le c_3 \frac{(r_l - r_{l-1})^{\frac{\theta_l}{2}}}{\sqrt{\theta_l}} \frac{1}{\sqrt{N}} \le \frac{c_3}{\sqrt{\theta_l}} \frac{\sqrt{2(r_l - t)^{\theta_l}}}{\sqrt{n}}.$$

The case $n = 1$ implies that $t_{k-1}^{N,\theta_l} \le t < t_k^{N,\theta_l} = r_l$. As in $(C3_l) \Longrightarrow (C4_l)$ we have

$$\|Y_{r_l} - Y_s\|_p \le c_3 \sqrt{\frac{1}{\theta_l}} (r_l - s)^{\frac{\theta_l}{2}} \le c_3 \sqrt{\frac{1}{\theta_l}} (r_l - t)^{\frac{\theta_l}{2}}$$

for all $s \in [t, r_l]$ so that

$$e_1((Y_s)_{s \in [t, r_l]} | L_p) \le c_3 \sqrt{\frac{1}{\theta_l}} (r_l - t)^{\frac{\theta_l}{2}}.$$

$\square$

3.2 Proof of Theorem 2.5

(a) We get, a.s., that

$$\begin{aligned} X_s^\eta - X_s = & \int_0^s [b(r, X_r^\eta) - b(r, X_r)] dr \\ & + \int_0^s [\sigma(r, X_r^\eta) - \sigma(r, X_r)] \sqrt{1 - \eta(r)^2} dW_r \\ & + \int_0^s \sigma(r, X_r^\eta) \eta(r) dB_r \\ & - \int_0^s \sigma(r, X_r)(1 - \sqrt{1 - \eta(r)^2}) dW_r. \end{aligned}$$

Using the Burkholder-Davies-Gundy inequalities we estimate

$$e(s) := \mathbb{E} \sup_{0\le r\le s} |X^\eta_r - X_r|^p$$

by

$$\begin{aligned} e(s) &\le& 4^{p-1}\Big[T^{p-1}L_b^p \int_0^s e(r)dr + a_p^p T^{p/2-1} L_\sigma^p \int_0^s e(r)dr \\ && + a_p^p \|\sigma\|_\infty^p \left(\int_0^s \eta(r)^2 dr\right)^{\frac{p}{2}} + a_p^p \|\sigma\|_\infty^p \left(\int_0^s (1-\sqrt{1-\eta(r)^2})^2 dr\right)^{\frac{p}{2}}\Big], \end{aligned}$$

where L_b and L_σ are the Lipschitz constants (with respect to x) of b and σ, and a_p the constant from the Burkholder-Davis-Gundy inequality. Note that $1 - \sqrt{1-\eta(r)^2} = \frac{\eta(r)^2}{1+\sqrt{1-\eta(r)^2}} \le |\eta(r)|$ using $|\eta(r)| \le 1$. Thus, applying Gronwall's lemma implies

$$\left\| \sup_{0\le r\le s} |X^\eta_r - X_r| \right\|_p \le c_{(21)} \left(\int_0^s \eta(r)^2 dr\right)^{\frac{1}{2}} \tag{21}$$

where $c_{(21)} > 0$ depends at most on (p, T, b, σ).

(b) We consider $Y^\eta - Y$ and $Z^\eta - Z$ and relate (Y, Z) and (Y^η, Z^η) to two BSDEs driven by the same Brownian motion (W, B). This is the purpose of the construction below.

Let $\varphi = \chi_{[-1/2,1/2]}$ so that

$$\sup_{\eta\in[-1,1]} \left(\frac{\varphi(\eta)}{\sqrt{1-\eta^2}} + \frac{1-\varphi(\eta)}{|\eta|} \right) = 2 \tag{22}$$

using the convention $\frac{0}{0} = 0$. Thus, we can define the parameterized driver

$$f^\eta(t, \omega, y, z) := f\left(t, X^\eta_t(\omega), y, z^W \frac{\varphi(\eta(t))}{\sqrt{1-\eta(t)^2}} + z^B \frac{1-\varphi(\eta(t))}{\eta(t)}\right)$$

where $z = (z^W, z^B)$ is $2d$-dimensional. In view of (22), the driver f^η is Lipschitz with respect to y and z. Thus, for any $\mathcal{F}^{W,B}_T$-measurable terminal condition $\widetilde{\xi} \in L_p$, there is an unique solution in L_p in the filtration $\mathcal{F}^{W,B}$ to the BSDE

$$\widetilde{Y}_t = \widetilde{\xi} + \int_t^T f^\eta(s, \widetilde{Y}_s, \widetilde{Z}_s)ds - \int_t^T \widetilde{Z}^W_s dW_s - \int_t^T \widetilde{Z}^B_s dB_s$$

because of [6, Theorem 4.2].

(c) For the driver f^0 (i.e. $\eta \equiv 0$) and terminal condition ξ we have that

$$(Y, [Z, 0])$$

solves our BSDE.
(d) For the driver f^η and the terminal condition ξ^η we have that

$$(Y^\eta, [Z^{\eta,W}, Z^{\eta,B}])$$

with

$$Z_s^{\eta,W} = Z_s^\eta \sqrt{1-\eta(s)^2} \quad \text{and} \quad Z_s^{\eta,B} = Z_s^\eta \eta(s)$$

solves our BSDE because

$$\begin{aligned} Y_t^\eta = \xi^\eta &+ \int_t^T f(s, X_s^\eta, Y_s^\eta, Z_s^\eta \varphi(\eta(s)) + Z_s^\eta(1-\varphi(\eta(s))))ds \\ &- \int_t^T Z_s^\eta \sqrt{1-\eta(s)^2} dW_s - \int_t^T Z_s^\eta \eta(s) dB_s \\ = \xi^\eta &+ \int_t^T f^\eta(s, Y_s^\eta, [Z_s^{\eta,W}, Z_s^{\eta,B}])ds - \int_t^T Z_s^{\eta,W} dW_s - \int_t^T Z_s^{\eta,B} dB_s. \end{aligned}$$

(e) To sum up, we have proved that $(Y, [Z, 0])$ and $(Y^\eta, [Z^\eta_. \sqrt{1-\eta(.)^2}, Z^\eta_. \eta(.)])$ solve the BSDEs with data (ξ, f^0) and (ξ^η, f^η) in the filtration $(\mathcal{F}_t^{W,B})_{t\in[0,T]}$. Then, we are in a position to apply Lemma A.1 (with d replaced by $2d$) and get

$$\begin{aligned} &\left\| \sup_{0\le t\le T} |Y_t^\eta - Y_t| \right\|_p + \left\| \left(\int_0^T \left| (Z_t^\eta \sqrt{1-\eta(t)^2} - Z_t, Z_t^\eta \eta(t)) \right|^2 dt \right)^{1/2} \right\|_p \\ \le\ & c_{(A.1)} \left(\|\xi^\eta - \xi\|_p + \left\| \int_0^T |f^\eta(t, Y_t, [Z_t, 0]) - f^0(t, Y_t, [Z_t, 0])| dt \right\|_p \right) \\ \le\ & c_{(A.1)} \Bigg(\|\xi^\eta - \xi\|_p + L_f \left\| \int_0^T |X_t^\eta - X_t| dt \right\|_p \\ & \qquad + L_f \left\| \int_0^T |Z_t| \left| \frac{\varphi(\eta(t))}{\sqrt{1-\eta(t)^2}} - 1 \right| dt \right\|_p \Bigg) \end{aligned}$$

where $c_{(A.1)}$ (here and thereafter) is not identical with the constant c in Lemma A.1 but only refers to the fact that the inequality of Lemma A.1 is used. Now, since $\left| \frac{\varphi(\eta)}{\sqrt{1-\eta^2}} - 1 \right| \le c_\varphi |\eta|$ for some constant $c_\varphi > 0$, we have

$$\int_0^T |Z_t| \left| \frac{\varphi(\eta(t))}{\sqrt{1-\eta(t)^2}} - 1 \right| dt \le c_\varphi \left(\int_0^T |Z_t|^2 dt \right)^{1/2} \left(\int_0^T \eta(t)^2 dt \right)^{1/2}.$$

With the previous estimate on $X^\eta - X$ from (21) this leads to

$$\left\| \sup_{0\le t\le T} |Y_t^\eta - Y_t| \right\|_p + \left\| \left(\int_0^T \left| (Z_t^\eta \sqrt{1-\eta(t)^2} - Z_t, Z_t^\eta \eta(t)) \right|^2 dt \right)^{\frac{1}{2}} \right\|_p$$

$$
\begin{aligned}
\leq\;& c_{(A.1)}\Bigg[\|\xi^\eta-\xi\|_p \\
&+L_f\left[Tc_{(21)}+c_\varphi\left\|\left(\int_0^T|Z_t|^2dt\right)^{\frac{1}{2}}\right\|_p\right]\left(\int_0^T\eta(t)^2dt\right)^{\frac{1}{2}}\Bigg].
\end{aligned}
$$

Applying Lemma A.1 to $\xi^{(0)}=0$, $f_0=0$, $Y_s^{(0)}\equiv 0$, $Z_s^{(0)}\equiv 0$, $\xi^{(1)}=\xi$, $f_1(\omega;s,y,z):=f(s,X_s(\omega),y,z)$ and our solution (Y,Z) we obtain

$$
\alpha_s(\omega)=|f(s,X_s(\omega),0,0)|\leq K_f+L_f\sup_{0\leq t\leq T}|X_t(\omega)|
$$

and

$$
\left\|\left(\int_0^T|Z_t|^2dt\right)^{\frac{1}{2}}\right\|_p\leq c_{(A.1)}[K_f+L_f+\|\xi\|_p].
$$

To complete the proof, it remains to use the inequality

$$
\begin{aligned}
|(Z_t^\eta\sqrt{1-\eta(t)^2}-Z_t,Z_t^\eta\eta(t))|^2 &= |Z_t^\eta|^2+|Z_t|^2-2\sqrt{1-\eta(t)^2}\langle Z_t^\eta,Z_t\rangle \\
&\geq \frac{1}{2}|Z_t^\eta-Z_t|^2.
\end{aligned}
$$

□

3.3 Proof of Theorem 2.10

(a) In this step we assume that all $(x_1,...,x_L)$ (and similarly $(x_1',...,x_L')$) that appear have the property that $x_1\in D_2$, $(x_1,x_2)\in D_3$, ..., $(x_1,...,x_{L-1})\in D_L$ where the sets $D_2,...,D_L$ are taken from Proposition 1.1. By backward induction we prove the following estimate regarding the terminal condition function $\Phi_l(x_1,...,x_l):=u_l(x_1,...,x_{l-1};r_l,x_l)$ of the BSDE at time r_l:

$$
|\Phi_l(\overline{x}_l)-\Phi_l(\overline{x}_l')|\leq c_l\sum_{i=1}^{l}\left[|g_i(x_i)-g_i(x_i')|+\psi_i(\overline{x}_i;\overline{x}_i')|x_i-x_i'|\right]. \qquad (23)
$$

This is true for $l=L$ by our assumption. Assume now that (23) holds for some $2\leq l\leq L$ and let us prove the inequality for $l-1$. We have

$$
\begin{aligned}
&\left|\Phi_{l-1}(x_1,...,x_{l-1})-\Phi_{l-1}(x_1',...,x_{l-1}')\right| \\
\leq\;& \left|u_l(x_1,...,x_{l-1};r_{l-1},x_{l-1})-u_l\left(x_1',...,x_{l-1}';r_{l-1},x_{l-1}\right)\right| \\
&+\left|u_l\left(x_1',...,x_{l-1}';r_{l-1},x_{l-1}\right)-u_l\left(x_1',...,x_{l-1}';r_{l-1},x_{l-1}'\right)\right| \\
\leq\;& \left|u_l(x_1,...,x_{l-1};r_{l-1},x_{l-1})-u_l\left(x_1',...,x_{l-1}';r_{l-1},x_{l-1}\right)\right| \\
&+\frac{\alpha_l}{\sqrt{r_l-r_{l-1}}} \\
&(1+|x_1'|^{q_{l,1}}+\cdots+|x_{l-1}'|^{q_{l,l-1}}+|x_{l-1}|^{q_{l,l}}+|x_{l-1}'|^{q_{l,l}})|x_{l-1}-x_{l-1}'|
\end{aligned}
$$

where we used Proposition 1.1. To estimate the remaining first term we use Lemma A.1 and get that

$$\begin{aligned}
& \left|u_l\left(x_1,...,x_{l-1};r_{l-1},x_{l-1}\right)-u_l\left(x_1',...,x_{l-1}';r_{l-1},x_{l-1}\right)\right| \\
\leq\ & c_{(A.1)}\|u_l\left(x_1,...,x_{l-1};r_l,X_{r_l}^{r_{l-1},x_{l-1}}\right)-u_l\left(x_1',...,x_{l-1}';r_l,X_{r_l}^{r_{l-1},x_{l-1}}\right)\|_2 \\
=\ & c_{(A.1)}\|\Phi_l\left(x_1,...,x_{l-1},X_{r_l}^{r_{l-1},x_{l-1}}\right)-\Phi_l\left(x_1',...,x_{l-1}',X_{r_l}^{r_{l-1},x_{l-1}}\right)\|_2 \\
\leq\ & c_{(A.1)}c_{(23)}\Big(\sum_{i=1}^{l-1}[|g_i(x_i)-g_i(x_i')|+\psi_i(x_1,...,x_i;x_1',...,x_i')|x_i-x_i'|]\Big).
\end{aligned}$$

(b) In the second step we verify the fractional smoothness, where we use (4) and therefore the inequalities from step (a). For $r_{l-1}\leq s<r_l$, we have

$$\|Y_{r_l}-\mathbb{E}(Y_{r_l}|\mathcal{F}_s)\|_p=\|\Phi_l(X_{r_1},...,X_{r_l})-\mathbb{E}(\Phi_l(X_{r_1},...,X_{r_l})|\mathcal{F}_s)\|_p.$$

In particular, this expression depends on x_0, b, σ, $r_1,...,r_l$, s and Φ_l but not on the specific realization of the diffusion X. Hence we can assume the extended setting from Section 2.2.1. Using inequalities (6) and the estimate (23) implies that

$$\begin{aligned}
& \|\Phi_l(X_{r_1},...,X_{r_l})-\mathbb{E}(\Phi_l(X_{r_1},...,X_{r_l})|\mathcal{F}_s)\|_p \\
\leq\ & \|\Phi_l(X_{r_1},...,X_{r_l})-\Phi_l(X_{r_1},...,X_{r_{l-1}},X_{r_l}^{\eta_{s,r_l}})\|_p \\
\leq\ & c_l\Bigg\|\left|g_l(X_{r_L})-g_l(X_{r_l}^{\eta_{s,r_l}})\right| \\
& +\psi_l(X_{r_1},...,X_{r_l};X_{r_1},...,X_{r_{l-1}},X_{r_l}^{\eta_{s,r_l}})|X_{r_l}-X_{r_l}^{\eta_{s,r_l}}|\Bigg\|_p \\
\leq\ & c_l\|g_l(X_{r_l})-g_l(X_{r_l}^{\eta_{s,r_l}})\|_p \\
& +c_l\|\psi_l(X_{r_1},...,X_{r_l};X_{r_1},...,X_{r_{l-1}},X_{r_l}^{\eta_{s,r_l}})\|_{2p}\|X_{r_l}-X_{r_l}^{\eta_{s,r_l}}\|_{2p} \\
\leq\ & 2c_l\|g_l(X_{r_l})-\mathbb{E}(g_l(X_{r_l})|\mathcal{F}_s)\|_p \\
& +c_l\sup_{r_{l-1}\leq u\leq r_l}\|\psi_l(X_{r_1},...,X_{r_l};X_{r_1},...,X_{r_{l-1}},X_{r_l}^{\eta_{u,r_l}})\|_{2p}c_{(21)}\sqrt{r_l-s}.
\end{aligned}$$

□

4 Perspectives

As natural steps, which could follow this paper, we see the investigation of more sufficient conditions for the fractional smoothness of a BSDE and the investigation of the limiting case as the number of points $r_1,...,r_L$ tends to infinity. In this connection the question, to what extend the generator might be path-dependent, is of interest as well. Moreover, the investigation of the above results in the context of other types of BSDEs (for example including reflection) and the development of numerical algorithms based on the discretizations proposed in this paper would be important.

A Some lemmas about BSDEs

We fix a complete probability space $(M, \Sigma, \mathbb{Q})$, $0 \le r < R \le T$ (the upper bound T is used to bound some constants independently from R), $d \ge 1$ and a d-dimensional standard Brownian motion $B = (B_t)_{t \in [r,R]}$ with $B_r \equiv 0$. Furthermore, we assume that $(\mathcal{G}_t)_{t \in [r,R]}$ is the augmentation of the natural filtration of B. The diffusion $(X_s)_{s \in [r,R]}$ is considered with respect to the same σ and b as used before, restricted to the corresponding time interval. Regarding the flow $(X_s^{t,x})_{s,t \in [r,R], x \in \mathbb{R}^d}$ and the filtrations $(\mathcal{G}_s^t)_{s \in [t,R]}$ we use the same convention as in Section 1.1.

Lemma A.1 (L_p-stability of solutions of BSDEs)**.** *Let $2 \le p < \infty$, $f_i : M \times [r,R] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$ be measurable with respect to $Prog(M \times [r,R]) \times \mathcal{B}(\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^{k \times d})$ with $Prog(M \times [r,R])$ being the σ-algebra of progressively measurable subsets, and assume that, a.s.,*

$$Y_t^{(i)} = \xi^{(i)} + \int_t^R f_i(s, Y_s^{(i)}, Z_s^{(i)}) ds - \int_t^R Z_s^{(i)} dB_s \quad \text{for } i = 0,1 \text{ and } r \le t \le R$$

with

$$\int_r^R |f_i(s, Y_s^{(i)}, Z_s^{(i)})| ds + \sup_{r \le t \le R} |Y_t^{(i)}| + \left(\int_r^R |Z_s^{(i)}|^2 ds \right)^{\frac{1}{2}} \in L_p.$$

Let

$$\alpha_s(\omega) := |f_1(\omega; s, Y_s^{(0)}(\omega), Z_s^{(0)}(\omega)) - f_0(\omega; s, Y_s^{(0)}(\omega), Z_s^{(0)}(\omega))|$$

and suppose that there is a $L_{f_1} > 0$ such that

$$|f_1(\omega; s, u_1, v_1) - f_1(\omega; s, u_2, v_2)| \le L_{f_1}[|u_1 - u_2| + |v_1 - v_2|].$$

Then there exists a $c_p > 0$, depending on p only, such that for $a \ge L_{f_1} + L_{f_1}^2$ one has

$$\mathbb{E}\left[\sup_{t \in [r,R]} e^{ap(t-r)} |\Delta Y_t|^p + \left(\int_r^R e^{2a(s-r)} |\Delta Z_s|^2 ds \right)^{\frac{p}{2}} \right] \le c_p^p \mathbb{E}\left[e^{ap(R-r)} |\Delta \xi|^p + \left(\int_r^R e^{a(s-r)} \alpha_s ds \right)^p \right].$$

Proof. The result is a direct consequence of [6, Proposition 3.2]. For $\Delta Y_t := Y_t^1 - Y_t^0$, $\Delta Z_t := Z_t^1 - Z_t^0$ and $\Delta \xi := \xi^{(1)} - \xi^{(0)}$ we get that

$$\Delta Y_t = \Delta \xi + \int_t^R \widehat{f}(s, \Delta Y_s, \Delta Z_s) ds - \int_t^R \Delta Z_s dB_s$$

with $\widehat{f}(s, \Delta y, \Delta z) := f_1(s, \Delta y + Y_s^{(0)}, \Delta z + Z_s^{(0)}) - f_0(s, Y_s^{(0)}, Z_s^{(0)})$ and

$$|\widehat{f}(\omega; s, \Delta y, \Delta z)|$$

$$
\begin{aligned}
&= |f_1(\omega; s, \Delta y + Y_s^{(0)}(\omega), \Delta z + Z_s^{(0)}(\omega)) - f_0(\omega; s, Y_s^{(0)}(\omega), Z_s^{(0)}(\omega))| \\
&\leq |f_1(\omega; s, Y_s^{(0)}(\omega), Z_s^{(0)}(\omega)) - f_0(\omega; s, Y_s^{(0)}(\omega), Z_s^{(0)}(\omega))| \\
&\quad + |f_1(\omega; s, \Delta y + Y_s^{(0)}(\omega), \Delta z + Z_s^{(0)}(\omega)) - f_1(\omega; s, Y_s^{(0)}(\omega), Z_s^{(0)}(\omega))| \\
&\leq \alpha_s(\omega) + L_{f_1}[|\Delta y| + |\Delta z|].
\end{aligned}
$$

Applying [6, Proposition 3.2] implies the assertion. □

The following lemma shows that [24, Theorem 3.2] transfers to our path dependent setting as expected. The proof is presumably only included in this preprint version for the convenience of the reader as it is a copy of that one in [24] (see also [19, Section 5]).

Lemma A.2 (Representation of a BSDE parameterized by a parameter $y \in \mathbb{R}^K$). *Assume that $(A_{b,\sigma})$ and (A_f) are satisfied, that $K, d \geq 1$ and that $H : \mathbb{R}^K \times \mathbb{R}^d \to \mathbb{R}$ is Borel-measurable with*

$$|H(y; x)| \leq \alpha(1 + |y|^\gamma + |x|^\beta) =: \psi(y, x)$$

for some $\alpha, \beta, \gamma \in [1, \infty)$. Then there exists a Borel set $F \subseteq \mathbb{R}^K$ such that F^c is of Lebesgue measure zero and such that for

$$G(y; x) := \chi_F(y) H(y; x)$$

and

$$U(y; t, x) := \begin{cases} Y_t^{y;t,x} \text{ a.s.} & : \quad r \leq t < R \\ G(y; x) & : \quad t = R \end{cases},$$

where $(Y_s^{y;t,x})_{s \in [t,R]}$ is the Y-component of the BSDE with respect to the forward diffusion $(X_s^{t,x})_{s \in [t,R]}$, the terminal condition $G(y; X_R^{t,x})$ with terminal time $R \in (0, T]$, and the generator f, the following assertions are satisfied:

(i) *For fixed $y \in \mathbb{R}^K$ we have that $U(y; \cdot, \cdot) \in C^{0,1}([r, R) \times \mathbb{R}^d)$.*

(ii) *The functions $U : \mathbb{R}^K \times [r, R] \times \mathbb{R}^d \to \mathbb{R}$ and $\nabla_x U : \mathbb{R}^K \times [r, R) \times \mathbb{R}^d \to \mathbb{R}^{1 \times d}$ are measurable.*

(iii) *There exists a constant $c > 0$ depending at most on $(b, \sigma, T, \alpha, \gamma, \beta, K_f, L_f)$ such that*

  (a) $|U(y; t, x)| \leq c\psi(y; x)$ *for* $(y, t, x) \in \mathbb{R}^K \times [r, R] \times \mathbb{R}^d$,

  (b) $|\nabla_x U(y; t, x)| \leq c\frac{\psi(y;x)}{\sqrt{R-t}}$ *for* $(y, t, x) \in \mathbb{R}^K \times [r, R) \times \mathbb{R}^d$.

(iv) *For any $y \in \mathbb{R}^K$, the solution of the BSDE with the terminal condition $G(y; X_R^{r,x})$, generator f, and forward diffusion $(X_s^{r,x})_{s \in [r,R]}$ can be represented as*

  (a) $Y_t^{y;r,x} = U(y; t, X_t^{r,x})$ *on* $[r, R]$,

  (b) $Z_t^{y;r,x} = \nabla_x U(y; t, X_t^{r,x})\sigma(t, X_t^{r,x})$ *on* $[r, R)$.

Proof. We find $H_n \in C_0^\infty(\mathbb{R}^K \times \mathbb{R}^d)$, $n \geq 1$, such that

$$\lim_n H_n = H \quad \lambda_{K+d}\text{-a.e.} \quad \text{and} \quad |H_n(y;x)| \leq 2\psi(y,x).$$

Hence there is a Borel set $F \subseteq \mathbb{R}^K$ such that F^c is of Lebesgue measure zero and such that

$$\lim_n G_n(y;\cdot) = G(y;\cdot) \quad \lambda_d\text{-a.e.}$$

for all $y \in \mathbb{R}^K$ with

$$G_n(y;x) := \chi_F(y) H_n(y;x) \quad \text{and} \quad G(y;x) := \chi_F(y) H(y;x).$$

Let U^n be defined as U with G replaced by G_n. Applying [20, Theorems 3.1 and 4.2] gives that

$$(U^n(y;s,X_s^{t,x}), \nabla_x U^n(y;s,X_s^{t,x})\sigma(s,X_s^{t,x}))_{s\in[t,R]}$$

solves our BSDE on the interval $[t,R]$, that $U^n(y;\cdot,\cdot) \in C^{0,1}([r,R]\times\mathbb{R}^d)$ and that

$$\nabla_x U^n(y;t,x) = \\ \mathbb{E}\left[G_n(y;X_R^{t,x}) N_R^{t,1,(t,x)} + \int_t^R f(s,X_s^{t,x},Y_s^{y,n;t,x},Z_s^{y,n;t,x}) N_s^{t,1,(t,x)} ds \right].$$

Properties of the function U^n

(a) To estimate $U^n(y;t,x)$ we let $U_0^n(y;t,x)$ be the corresponding solution with the zero generator and denote by $(Y_{s,0}^{y,n;t,x}, Z_{s,0}^{y,n;t,x})_{s\in[t,R]}$ the corresponding solution to our BSDE. By Lemma A.1 we get that

$$\begin{aligned}
& |U^n(y;t,x) - U_0^n(y;t,x)| \\
\leq\ & c_{(A.1)} \left\| \int_t^R |f(s,X_s^{t,x},Y_{s,0}^{y,n;t,x},Z_{s,0}^{y,n;t,x})| ds \right\|_2 \\
\leq\ & c_{(A.1)} K_f R + c_{(A.1)} L_f \left\| \int_t^R [|X_s^{t,x}| + |Y_{s,0}^{y,n;t,x}| + |Z_{s,0}^{y,n;t,x}|] ds \right\|_2 \\
\leq\ & c_{(A.1)} K_f R + c_{(A.1)} L_f \times \\
& \left\| \int_t^R [|X_s^{t,x}| + |\mathbb{E}(G_n(y;X_R^{t,x})|\mathcal{G}_s^t)| + \|\sigma\|_\infty |\mathbb{E}(G_n(y;X_R^{t,x}) N_R^{s,1,(t,x)}|\mathcal{G}_s^t)] ds \right\|_2 \\
\leq\ & c_{(A.1)} K_f R + c_{(A.1)} L_f \times \\
& \left\| \int_t^R [|X_s^{t,x}| + 2|\mathbb{E}(\psi(y;X_R^{t,x})^2|\mathcal{G}_s^t)|^{\frac{1}{2}} (1 + \|\sigma\|_\infty |\mathbb{E}((N_R^{s,1,(t,x)})^2|\mathcal{G}_s^t)|^{\frac{1}{2}})] ds \right\|_2 .
\end{aligned}$$

Now we use that

$$\mathbb{E}((N_R^{s,1,(t,x)})^2|\mathcal{G}_s^t) \leq \frac{\kappa_2^2}{R-s} \quad \text{and} \quad |U_0^n(y;t,x)| \leq 2\mathbb{E}\psi(y;X_R^{t,x})$$

and

$$\|\psi(y;X_s^{t,x})\|_2 \le \beta_2 \psi(y;x) \quad \text{and} \quad \|X_s^{t,x}\|_2 \le \alpha_2[1+|x|]$$

to get

$$\begin{aligned}
& |U^n(y;t,x)| \\
\le\ & 2\mathbb{E}\psi(y;X_R^{t,x}) + c_{(A.1)}K_f R + c_{(A.1)}L_f \times \\
& \left\| \int_t^R [|X_s^{t,x}| + 2|\mathbb{E}(\psi(y;X_R^{t,x})^2|\mathcal{G}_s^t)|^{\frac{1}{2}}(1+\|\sigma\|_\infty \kappa_2 (R-s)^{-1/2})]ds \right\|_2 \\
\le\ & 2\mathbb{E}\psi(y;X_R^{t,x}) + c_{(A.1)}K_f R + c_{(A.1)}L_f \times \\
& \left[\int_t^R \|X_s^{t,x}\|_2 ds + 2\int_t^R [\|\psi(y;X_R^{t,x})\|_2 (1+\|\sigma\|_\infty \kappa_2 (R-s)^{-1/2}) ds \right] \\
\le\ & \|\psi(y;X_R^{t,x})\|_2 \left[2 + 2c_{(A.1)}L_f[R+2\|\sigma\|_\infty \kappa_2 R^{1/2}] \right] \\
& + c_{(A.1)}L_f \int_t^R \|X_s^{t,x}\|_2 ds + c_{(A.1)}K_f R \\
\le\ & \beta_2 \psi(y;x) \left[2 + 2c_{(A.1)}L_f[R+2\|\sigma\|_\infty \kappa_2 R^{1/2}] \right] \\
& + c_{(A.1)}L_f \int_t^R \alpha_2[1+|x|] ds + c_{(A.1)}K_f R
\end{aligned}$$

so that, for some $c_{(24)} \ge 1$,

$$|U^n(y;t,x)| \le c_{(24)}\psi(y;x). \tag{24}$$

(b) According to [20, Theorem 3.1, Corollary 3.2] the gradient $\nabla_x U^n(y;t,x)$ exists and is bounded by a constant that might depend on n and y. For $r \le t \le \rho \le R$ and $y \in \mathbb{R}^K$ we define

$$\begin{aligned}
A_t^\rho &:= \sqrt{\rho - t} \sup_{x\in\mathbb{R}^d} \frac{|\nabla_x U^n(y;t,x)|}{\psi(y;x)}, \\
B_t^\rho &:= \sup_{s\in[t,\rho]} A_s^\rho.
\end{aligned}$$

Although A^ρ and B^ρ might depend on n and y, we do not indicate this for the purpose of notational simplicity. Using [20, Theorem 4.2] yields

$$\begin{aligned}
& |\nabla_x U^n(y;t,x)| \\
=\ & \left| \mathbb{E}\left[U^n(y;\rho,X_\rho^{t,x}) N_\rho^{t,1,(t,x)} + \int_t^\rho f(s,X_s^{t,x},Y_s^{y,n;t,x},Z_s^{y,n;t,x}) N_s^{t,1,(t,x)} ds \right] \right| \\
\le\ & c_{(24)}\kappa_2 \frac{\|\psi(y;X_\rho^{t,x})\|_2}{\sqrt{\rho-t}} + \kappa_2 \int_t^\rho \frac{\|f(s,X_s^{t,x},Y_s^{y,n;t,x},Z_s^{y,n;t,x})\|_2}{\sqrt{s-t}} ds \\
\le\ & c_{(24)}\kappa_2 \frac{\|\psi(y;X_\rho^{t,x})\|_2}{\sqrt{\rho-t}} + \kappa_2 \times
\end{aligned}$$

$$
\begin{aligned}
&\int_t^\rho \frac{K_f + L_f[\|X_s^{t,x}\|_2 + c_{(24)}\|\psi(y;X_s^{t,x})\|_2 + \|\nabla_x U^n(y;s,X_s^{t,x})\sigma(s,X_s^{t,x})\|_2]}{\sqrt{s-t}} ds \\
&\leq \quad c_{(24)}\kappa_2 \frac{\|\psi(y;X_\rho^{t,x})\|_2}{\sqrt{\rho-t}} + \kappa_2 \times \\
&\quad \int_t^\rho \frac{K_f + L_f[\|X_s^{t,x}\|_2 + c_{(24)}\|\psi(y;X_s^{t,x})\|_2 + \|\sigma\|_\infty \|\nabla_x U^n(y;s,X_s^{t,x})\|_2]}{\sqrt{s-t}} ds \\
&\leq \quad c_{(24)}\kappa_2 \frac{\|\psi(y;X_\rho^{t,x})\|_2}{\sqrt{\rho-t}} + \\
&\quad \kappa_2 \int_t^\rho \frac{K_f + L_f\left[\|X_s^{t,x}\|_2 + c_{(24)}\|\psi(y;X_s^{t,x})\|_2 + \|\sigma\|_\infty \left\| A_s^\rho \frac{\psi(y;X_s^{t,x})}{\sqrt{\rho-s}}\right\|_2\right]}{\sqrt{s-t}} ds \\
&\leq \quad c_{(24)}\kappa_2 \frac{\|\psi(y;X_\rho^{t,x})\|_2}{\sqrt{\rho-t}} \\
&\qquad + \kappa_2 2\sqrt{\rho-r}[K_f + L_f + L_f c_{(24)}] \sup_{s\in[t,\rho]} [1 + \|X_s^{t,x}\|_2 + \|\psi(y;X_s^{t,x})\|_2] \\
&\qquad + \kappa_2 \|\sigma\|_\infty B_t^\rho \sup_{s\in[t,\rho]} \|\psi(y;X_s^{t,x})\|_2 B\left(\frac{1}{2},\frac{1}{2}\right) \\
&\leq \quad c_{(24)}\kappa_2 \frac{\beta_2\psi(y;x)}{\sqrt{\rho-t}} + \kappa_2 2\sqrt{\rho-r}[K_f + L_f + L_f c_{(24)}][1 + \alpha_2[1+|x|] \\
&\qquad + \beta_2\psi(y;x)] + \kappa_2\|\sigma\|_\infty B_t^\rho \beta_2 \psi(y;x) B\left(\frac{1}{2},\frac{1}{2}\right) \\
&\leq \quad A_\psi \psi(y;x)\left(\frac{1}{\sqrt{\rho-t}} + B_t^\rho\right)
\end{aligned}
$$

where $A_\psi > 0$ depends at most on $(b,\sigma,T,\alpha,\beta,\gamma,K_f,L_f)$. Consequently,

$$
A_t^\rho \leq A_\psi\left(1 + \sqrt{\rho-t} B_t^\rho\right)
$$

and

$$
B_t^\rho \leq A_\psi\left(1 + \sqrt{\rho-t} B_t^\rho\right).
$$

In case of $|\rho - t| \leq (2A_\psi)^{-2}$ this gives $B_t^\rho \leq 2A_\psi$ and

$$
|\nabla_x U^n(y;t,x)| \leq 2A_\psi \frac{\psi(y;x)}{\sqrt{\rho-t}}.
$$

Moreover, in case of $\frac{1}{4}(2A_\psi)^{-2} \leq |\rho - t| \leq (2A_\psi)^{-2}$ we also get that

$$
|\nabla_x U^n(y;t,x)| \leq 2A_\psi \frac{\psi(y;t,x)}{\sqrt{\rho-t}} \leq 8A_\psi^2 \psi(y;x).
$$

The latter inequality means that

$$
|\nabla_x U^n(y;t,x)| \leq 8A_\psi^2 \psi(y;x)
$$

whenever $r \leq t \leq R$ and $|R-t| \geq \frac{1}{4}(2A_\psi)^{-2}$. On the other side,

$$|\nabla_x U^n(y;t,x)| \leq 2A_\psi \frac{\psi(y;x)}{\sqrt{R-t}}$$

for $r \leq t \leq R$ and $|R-t| \leq (2A_\psi)^{-2}$. Combining both estimates yields to

$$|\nabla_x U^n(y;t,x)| \leq c_{(25)} \frac{\psi(y;x)}{\sqrt{R-t}} \tag{25}$$

for all $t \in [r,R]$.

(c) We show that $U^n(y;t,x)$ is measurable as function on $\mathbb{R}^K \times [r,R] \times \mathbb{R}^d$. Let $y, y' \in F$. Then it follows by Lemma A.1 that

$$\begin{aligned} |U^n(y;t,x) - U^n(y';t,x)| &\leq c_{(A.1)} \left\| G_n(y; X_R^{(t,x)}) - G_n(y'; X_R^{(t,x)}) \right\|_2 \\ &\leq c_{(A.1)} Lip(H^n)|y-y'| \end{aligned}$$

and

$$\begin{aligned} & |U^n(y;t,x) - U^n(y';t',x')| \\ \leq\ & |U^n(y;t,x) - U^n(y';t,x)| + |U^n(y';t,x) - U^n(y';t',x')| \\ \leq\ & c_{(A.1)} Lip(H_n)|y-y'| + |U^n(y';t,x) - U^n(y';t',x')|. \end{aligned}$$

Hence $(U^n)^{-1}(B) \cap (F \times [r,R] \times \mathbb{R}^d) \in \mathcal{B}(\mathbb{R}^K \times [r,R] \times \mathbb{R}^d)$ for all open sets $B \subseteq \mathbb{R}$. On F^c we have

$$(U^n)^{-1}(B) \cap (F^c \times [r,R] \times \mathbb{R}^d) = F^c \times \overline{U}^{-1}(B) \in \mathcal{B}(\mathbb{R}^K \times [r,R] \times \mathbb{R}^d)$$

where $\overline{U}(t,x)$ is the functional for the Y process with zero terminal condition. Consequently, U^n is measurable.

<u>Properties of the function U</u>

(d) Let D be the product of $[r,b] \subseteq [r,R)$ where $b \in (r,R)$ and a compact subset of $\mathbb{R}^d$. For $(t,x) \in D$ Lemma A.1 and Proposition B.1 yield

$$\begin{aligned} & |U(y;t,x) - U^n(y;t,x)|^2 \\ \leq\ & c_{(A.1)}^2 \|G(y;X_R^{t,x}) - G_n(y;X_R^{t,x})\|_2^2 \\ =\ & c_{(A.1)}^2 \int_{\mathbb{R}^d} \Gamma(t,x;R,\xi)|G(y;\xi) - G_n(y;\xi)|^2 d\xi \\ \leq\ & c_{(A.1)}^2 \int_{\mathbb{R}^d} c_{(B.1)} \gamma_{R-t}^d \left(\frac{x-\xi}{c_{(B.1)}} \right) |G(y;\xi) - G_n(y;\xi)|^2 d\xi \\ \leq\ & c_{(A.1)}^2 \int_{\mathbb{R}^d} \frac{c_{(B.1)}}{(2\pi(R-b))^{\frac{d}{2}}} e^{-\frac{1}{c_{(B.1)}^2} \frac{|x-\xi|^2}{(R-r)}} |G(y;\xi) - G_n(y;\xi)|^2 d\xi \\ \leq\ & c_{(A.1)}^2 \int_{\mathbb{R}^d} \frac{c_{(B.1)}}{(2\pi(R-b))^{\frac{d}{2}}} e^{\frac{1}{c_{(B.1)}^2} \frac{-(|\xi|^2/2)+|x|^2}{(R-r)}} |G(y;\xi) - G_n(y;\xi)|^2 d\xi. \end{aligned}$$

This implies that for all fixed parameters $y \in \mathbb{R}^K$ there is a uniform convergence on D of U^n towards U, so that $U(y;\cdot,\cdot)$ is continuous on $[r,R) \times \mathbb{R}^d$. Moreover, as limit of measurable functions U^n, the function $U : \mathbb{R}^K \times [r,R) \times \mathbb{R}^d \to \mathbb{R}$ is measurable as well. Because $U(y;R,x) = G(y;x)$ the function $U : \mathbb{R}^K \times [r,R] \times \mathbb{R}^d \to \mathbb{R}$ is measurable.

(e) Now we get

$$\begin{aligned}
& \|Y_t^{y;r,x} - U(y;t,X_t^{r,x})\|_2 \\
\leq \; & \|Y_t^{y;r,x} - U^n(y;t,X_t^{r,x})\|_2 + \|U^n(y;t,X_t^{r,x}) - U(y;t,X_t^{r,x})\|_2 \\
\leq \; & c_{(A.1)}\|G(y;X_R^{r,x}) - G_n(y;X_R^{r,x})\|_2 + \|U^n(y;t,X_t^{r,x}) - U(y;t,X_t^{r,x})\|_2
\end{aligned}$$

where we have used Lemma A.1. Using dominated convergence both terms converge to zero as $n \to \infty$, because

$$|U(y;t,x)| \leq c_{(24)}\psi(y;x)$$

as a consequence of (24) and step (d). Consequently,

$$Y_t^{y;r,x} = U(y;t,X_t^{r,x}) \quad a.s. \quad \text{for all} \quad t \in [r,R].$$

<u>The function $\nabla_x U(y;t,x)$</u>

(f) By Lemma A.1 we know that

$$\begin{aligned}
& \lim_n \int_t^R \left\|\nabla_x U^n(y;s,X_s^{t,x})\sigma(s,X_s^{t,x}) - Z_s^{y;t,x}\right\|_2^2 ds \\
= \; & \lim_n \int_t^R \left\|Z_s^{y,n;t,x} - Z_s^{y;t,x}\right\|_2^2 ds = 0.
\end{aligned}$$

Let

$$V(y;t,x) := \mathbb{E}\left[G(y;X_R^{t,x})N_R^{t,1,(t,x)} + \int_t^R f(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x})N_s^{t,1,(t,x)}ds\right].$$

By dominated convergence we have that

$$\begin{aligned}
& \lim_n \nabla_x U^n(y;t,x) \\
= \; & \lim_n \mathbb{E}\Big[G_n(y;X_R^{t,x})N_R^{t,1,(t,x)} \\
& + \int_t^R f\big(s,X_s^{t,x},U^n(y;s,X_s^{t,x}),\nabla_x U^n(y;s,X_s^{t,x})\sigma(s,X_s^{t,x})\big)N_s^{t,1,(t,x)}ds\Big] \\
= \; & \mathbb{E}\Big[G(y;X_R^{t,x})N_R^{t,1,(t,x)} + \int_t^R f\big(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x}\big)N_s^{t,1,(t,x)}ds\Big] \\
= \; & V(y;t,x)
\end{aligned}$$

for all $(t,x) \in [r,R) \times \mathbb{R}^d$, which also implies

$$|V(y;t,x)| \leq c_{(25)}\frac{\psi(y;x)}{\sqrt{R-t}}$$

by (25). Consequently,

$$\lim_n \int_r^{R-\delta} \|\nabla_x U^n(y;t,X_t^{r,x})\sigma(t,X_t^{r,x}) - V(y;t,X_t^{r,x})\sigma(t,X_t^{r,x})\|_2^2\,dt = 0$$

for all $\delta \in (0, R-r)$ and

$$Z_t^{y;r,x} = V(y;t,X_t^{r,x})\sigma(t,X_t^{r,x}) \quad a.s.$$

for almost every $t \in [r,R)$. So we can re-define

$$Z_t^{y;r,x} := V(y;t,X_t^{r,x})\sigma(t,X_t^{r,x}).$$

(g) Next we show that

$$V(y;t,x) = \mathbb{E}\left[G(y;X_R^{t,x})N_R^{t,1,(t,x)} + \int_t^R f(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x})N_s^{t,1,(t,x)}ds\right]$$

is continuous in (t,x) on $[r,R)\times\mathbb{R}^d$. For the first term this follows from the classical theory from the properties of the transition density of X because

$$\mathbb{E}\left[G(y;X_R^{t,x})N_R^{t,1,(t,x)}\right] = \int_{\mathbb{R}^d} G(y;w)\nabla_x\Gamma(t,x;R,w)dw$$

and the continuity in (t,x) follows from Proposition B.1. So it remains to show that

$$(t,x) \to \mathbb{E}\int_t^R f(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x})N_s^{t,1,(t,x)}ds$$

is continuous in (t,x) on each D which is the product of $[r,b]\subseteq[r,R)$ and a compact subset of $\mathbb{R}^d$. Take a sequence $(t_n,x_n)\to(t,x)$ from D. We write

$$\begin{aligned}
& \mathbb{E}\int_t^R f(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x})N_s^{t,1,(t,x)}ds \\
= & \int_{(r,R)} \frac{\chi_{(t,R)}(s)}{\sqrt{R-s}\sqrt{s-t}} \\
& \mathbb{E}\big[[\sqrt{R-s}f(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x})][\sqrt{s-t}N_s^{t,1,(t,x)}]\big]ds \\
= & \int_{(r,R)} \varphi_t(s)\psi_{t,x}(s)ds
\end{aligned}$$

with

$$\begin{aligned}
\varphi_t(s) &:= \frac{\chi_{(t,R)}(s)}{\sqrt{R-s}\sqrt{s-t}}, \\
\psi_{t,x}(s) &:= \chi_{(t,R)}(s)\mathbb{E}\big[[\sqrt{R-s}f(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x})][\sqrt{s-t}N_s^{t,1,(t,x)}]\big].
\end{aligned}$$

The family $(\varphi_t)_{t\in[r,b]}$ is uniformly integrable for $b\in[r,R)$. The boundedness of $(\psi_{t,x})_{(t,x)\in D}$ follows from

$$|\psi_{t,x}(s)| \leq \sqrt{R-s}\|f(s,X_s^{t,x},Y_s^{y;t,x},Z_s^{y;t,x})\|_2\sqrt{s-t}\|N_s^{t,1,(t,x)}\|_2$$

$$
\begin{aligned}
\leq \quad & \sqrt{R-s}\Bigg[K_f + L_f(\|X_s^{t,x}\|_2 + \|U(y;s,X_s^{t,x})\|_2 \\
& +\|\sigma\|_\infty \|V(y;s,X_s^{t,x})\|_2)\Bigg]\kappa_2
\end{aligned}
$$

and the previous estimates on U and V obtained by (24) and (25). Moreover,

$$
\lim_n \varphi_{t_n}(s) = \varphi_t(s) \quad \text{for all} \quad s \in (r,R)\backslash\{t\}.
$$

As for $\mathbb{E}[G(y;X_R^{t,x})N_R^{t,1,(t,x)}]$, we show that $\lim_n \psi_{t_n,x_n}(s) = \psi_{t,x}(s)$ for all $s \in (r,R) \setminus \{t\}$.

(h) Finally, we show that $\nabla_x U = V$. For $x_0, x_1 \in \mathbb{R}^d$ we have that

$$
U^n(y;t,x_0) - U^n(y;t,x_1) = \int_0^1 \langle \nabla_x U^n(y;t,x_0 + \lambda(x_1 - x_0)), x_1 - x_0\rangle d\lambda
$$

for $r \leq t < R$. By dominated convergence we have that

$$
U(y;t,x_0) - U(y;t,x_1) = \int_0^1 \langle V(y;t,x_0 + \lambda(x_1 - x_0)), x_1 - x_0\rangle d\lambda
$$

so that we are done. □

Lemma A.3 (L_p-bound for the Z-process for a singular generator)**.** *Assume condition $(A_{b,\sigma})$, $0 \leq r < R \leq T$, $2 \leq p < \infty$ and assume that $X = (X_s)_{s\in[r,R]}$ is the diffusion with parameters (b,σ) started in some $x_r \in \mathbb{R}^d$.* [1] *Consider the BSDE*

$$
U_t = \int_t^R h(s,X_s,U_s,V_s)ds - \int_t^R V_s dB_s \tag{26}
$$

with a generator $h : [r,R) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ which is measurable with respect to $\mathcal{B}([r,R)) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d\times d})$ and assume the following:

(i) *$h(s,\cdot,u,v)$ is continuous in x for fixed s,u,v.*

(ii) *$|h(s,x,u_1,v_1) - h(s,x,u_2,v_2)| \leq L(|u_1 - u_2| + |v_1 - v_2|)$ for some $L > 0$.*

(iii) *$|h(s,x,u,v)| \leq \alpha(s,x) + \lambda|u| + \mu|v|$ where $\alpha : [r,R) \times \mathbb{R}^d \to \mathbb{R}$ is non-negative and $\mathcal{B}([r,R)) \times \mathcal{B}(\mathbb{R}^d)$-measurable, $\alpha(s,\cdot)$ is continuous for fixed s and satisfies $\alpha(s,x) \leq \kappa(s)[1 + |x|^q]$ for some $q \geq 0$, where the function $\kappa(.) \geq 0$ is bounded on compact subintervals of $[r,R)$ and*

$$
\int_r^R \|\alpha(s,X_s)\|_p ds < \infty.
$$

[1] We would need to write X_s^{r,x_r} but use simply X_s to shorten the notation.

Then there exists an unique solution (U,V) such that

$$\sup_{r\le t\le R}|U_t| + \left(\int_r^R |V_t|^2 dt\right)^{\frac{1}{2}} \in L_p$$

and a constant $c = c(p,\sigma,b,T,L,\lambda,\mu) > 0$ such that

(1) $\|U_t\|_p \le c\|\int_t^R |\alpha(s,X_s)|ds\|_p$ *for $t \in [r,R)$,*

(2) *and there exists a Borel set $\mathcal{N} \subseteq [r,R)$ of Lebesgue measure zero such that*

$$\|V_t\|_p \le c\int_t^R \frac{\|\alpha(s,X_s)\|_p}{\sqrt{s-t}}ds$$

for all $t \in [r,R)\backslash\mathcal{N}$.

Proof. The local boundedness of κ ensures $\int_t^R \frac{\|\alpha(s,X_s)\|_p}{\sqrt{s-t}}ds < \infty$ for $t \in [r,R)$. The existence of the unique L_p-solution (U,V) follows from [6, Theorems 4.1 and 4.2] and the statement (1) follows from [6, Proposition 3.2] where we consider the BSDE with the generator $h^{(t)}(s,x,u,v) := h(s,x,u,v)$ if $s \in [t,R)$ and $h^{(t)} := 0$ otherwise, and the accordingly modified α. So we turn to the statement (2).

(a) Fix a bump-function $v : \mathbb{R}^d \to [0,\infty) \in C_0^\infty$ with $v(x) = 0$ for $|x| \ge 1$ and $\int_{\mathbb{R}^d} v(x)dx = 1$. For $N \ge 1$, $\varepsilon > 0$, $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}$ define

$$\begin{aligned} v_\varepsilon(x) &:= \frac{1}{\varepsilon^d} v\left(\frac{x}{\varepsilon}\right), \\ h_{\varepsilon,N}(s,x,u,v) &:= (v_\varepsilon^x * h^N)(s,x,u,v) \end{aligned}$$

where $h^N := (h_1^{N/\sqrt{d}},...,h_d^{N/\sqrt{d}})$ with $\xi^N = (\xi\wedge N)\vee(-N)$ for $\xi \in \mathbb{R}$ (so that $|h^N| \le N$) and the notation v_ε^x indicates that the convolution is taken with respect to x. Assumption (ii) implies that

$$|h_{\varepsilon,N}(s,x,u,v)| \le (v_\varepsilon^x * \alpha^N)(s,x) + \lambda|u| + \mu|v|.$$

The function $h_{\varepsilon,N}$ is uniformly Lipschitz in (x,u,v) as

$$\begin{aligned} &|h_{\varepsilon,N}(s,x_1,u_1,v_1) - h_{\varepsilon,N}(s,x_2,u_2,v_2)| \\ &\qquad \le L(|u_1-u_2| + |v_1-v_2|) + \sup_{s',x',u',v'} |\nabla_{x'} h_{\varepsilon,N}(s',x',u',v')|\,|x_1-x_2|, \end{aligned}$$

where we note that $\nabla_{x'} h_{\varepsilon,N}$ is a matrix, and

$$\begin{aligned} |\nabla_{x'} h_{\varepsilon,N}(s',x',u',v')| &= \varepsilon^{-d-1}\left|\int_{|\xi-x'|\le\varepsilon} (\nabla v)\left(\frac{x'-\xi}{\varepsilon}\right) h^N(s,\xi,u',v')d\xi\right| \\ &\le \varepsilon^{-1} vol(B_1(\mathbb{R}^d))N\|\nabla v\|_\infty. \end{aligned}$$

(b) Fix $N \geq 1$ and $\varepsilon > 0$, let

$$h_0(s,x,u,v) := h_{\varepsilon,N}(s,x,u,v)\chi_{[r,R)}(s)$$

and $\alpha_0(s,x) := (v_\varepsilon^x * \alpha^N)(s,x)\chi_{[r,R)}(s)$. Let (U^0, V^0) be the solution of our BSDE (26) with h replaced by h_0 according to [8, Theorem 2.6], where $U_s^0 := A^0(s, X_s)$ for a continuous and bounded function $A^0 : [r,R] \times \mathbb{R}^d \to \mathbb{R}^d$. It is also shown that $\lambda \times \mathbb{Q}(\{(t,\omega) \in [r,R] \times M : |V_s^0| > c\}) = 0$ for some $c > 0$. By considering a Picard iteration

$$U_t^{0,k} = \int_t^R h_0(s, X_s, A^0(s,X_s), V_s^{0,k-1})ds - \int_t^R V_s^{0,k} dB_s$$

with $U_s^{0,0} \equiv 0$ one can show by induction that $V_s^{0,k}$ can be realized as a measurable functional of s and X_s and obtains finally that there is a measurable function $B^0 : [r,R] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ with $\|B^0\|_\infty \leq c$, such that one can realize (using uniqueness from [8, Theorem 2.6]) V^0 as $V_s^0 = B^0(s,X_s)$. Now

$$h_0(s, X_s, U_s^0, V_s^0) = \mathbb{E}h_0(s, X_s, U_s^0, V_s^0) + \int_r^s \lambda_t^s dB_t \quad a.s.,$$

where the matrix λ_t^s is obtained via the PDE approach, so that we get, a.s.,

$$\begin{aligned} U_r^0 + \int_r^R V_t^0 dB_t &= \int_r^R h_0(s, X_s, U_s^0, V_s^0)ds \\ &= \int_r^R \mathbb{E}h_0(s, X_s, U_s^0, V_s^0)ds + \int_r^R \int_t^R \lambda_t^s ds dB_t \end{aligned}$$

by a stochastic Fubini argument and

$$V_t^0 = \int_t^R \lambda_t^s ds \quad a.s. \text{ for a.e. } t \in [r,R].$$

If the set of those t is denoted by $\mathcal{M}$, then for $t \in \mathcal{M}$,

$$\begin{aligned} \|V_t^0\|_p &\leq \int_t^R \|\lambda_t^s\|_p ds \\ &\leq \kappa_{p'} \int_t^R \frac{\|h_0(s, X_s, U_s^0, V_s^0)\|_p}{\sqrt{s-t}} ds \\ &\leq \kappa_{p'} \int_t^R \frac{\|a_0(s, X_s)\|_p + \lambda \|U_s^0\|_p + \mu \|V_s^0\|_p}{\sqrt{s-t}} ds \\ &= \kappa_{p'} \int_t^R \frac{\psi(s) + \mu \|V_s^0\|_p}{\sqrt{s-t}} ds \end{aligned}$$

with

$$\psi(s) := \|\alpha_0(s, X_s)\|_p + \lambda \|U_s^0\|_p.$$

Applying the same inequality to $s \in \mathcal{M}$ gives by iteration for $t \in \mathcal{M}$ that

$$\begin{aligned}
& \|V_t^0\|_p \\
\leq \quad & \kappa_{p'} \int_t^R \frac{\psi(s) + \mu\kappa_{p'} \int_s^R \frac{\psi(w)+\mu\|V_w^0\|_p}{\sqrt{w-s}} dw}{\sqrt{s-t}} ds \\
= \quad & \kappa_{p'} \int_t^R \frac{\psi(s)}{\sqrt{s-t}} ds + \mu\kappa_{p'}^2 B\left(\frac{1}{2}, \frac{1}{2}\right) \int_t^R \psi(s) ds + \\
& \qquad\qquad\qquad\qquad + (\mu\kappa_{p'})^2 B\left(\frac{1}{2}, \frac{1}{2}\right) \int_t^R \|V_s^0\|_p ds \\
\leq \quad & \left(\kappa_{p'} + \sqrt{T}\mu\kappa_{p'}^2 B\left(\frac{1}{2}, \frac{1}{2}\right)\right) \int_t^R \frac{\psi(s)}{\sqrt{s-t}} ds \\
& \qquad\qquad\qquad\qquad + (\mu\kappa_{p'})^2 B\left(\frac{1}{2}, \frac{1}{2}\right) \int_t^R \|V_s^0\|_p ds.
\end{aligned}$$

It follows from the boundedness properties of V_s^0 that $\int_r^R \|V_s^0\|_p ds < \infty$. For this reason we can apply Gronwall's lemma to derive

$$\|V_t^0\|_p \leq (\kappa_{p'} + \sqrt{T}\mu\kappa_{p'}^2 B(1/2, 1/2)) e^{(\mu\kappa_{p'})^2 B(1/2,1/2)(R-t)} \int_t^R \frac{\psi(s)}{\sqrt{s-t}} ds$$

for $t \in \mathcal{M}$. Next we estimate $\psi(s)$ by

$$\psi(s) \leq \|\alpha_0(s, X_s)\|_p + \lambda c_{(A.3)(1)} \int_s^R \|\alpha_0(w, X_w)\|_p dw,$$

where we use Lemma A.3(1) (with the same (L, λ, μ)), and get

$$\begin{aligned}
\|V_t^0\|_p \quad & \leq \quad d_1 \int_t^R \frac{\|\alpha_0(s, X_s)\|_p + \int_s^R \|\alpha_0(w, X_w)\|_p dw}{\sqrt{s-t}} ds \\
& \leq \quad d_1(1 + 2T) \int_t^R \frac{\|\alpha_0(s, X_s)\|_p}{\sqrt{s-t}} ds
\end{aligned}$$

with $d_1 := \left(\kappa_{p'} + \sqrt{T}\mu\kappa_{p'}^2 B\left(\frac{1}{2}, \frac{1}{2}\right)\right) e^{(\mu\kappa_{p'})^2 B\left(\frac{1}{2}, \frac{1}{2}\right)T}(1 + \lambda c_{(A.3)(1)})$. Hence, rewriting the dependence with respect to N and ϵ in our estimates, we have proved

$$\|V_t^{N,\varepsilon}\|_p \leq d_2 \int_t^R \frac{\|(v_\varepsilon^x * \alpha^N)(s, X_s)\|_p}{\sqrt{s-t}} ds \tag{27}$$

for $t \in \mathcal{M} = [r, R] \setminus \mathcal{N}_{N,\varepsilon}$ with $d_2 := d_1(1 + 2T)$.

(c) Let $\overline{\mathcal{N}} := \bigcup_{N,n} \mathcal{N}_{N,1/n}$ and let $((U_t^N, V_t^N))_{t \in [r,R]}$ be the solution of (26) with the generator h^N. Because

$$\lim_{\varepsilon \downarrow 0} \int_r^R \|h_{N,\varepsilon}(s, X_s, U_s^N, V_s^N) - h^N(s, X_s, U_s^N, V_s^N)\|_2 ds = 0$$

by dominated convergence (here we use the continuity of h in x) and

$$|h_{N,\varepsilon}(r,x,u_1,v_1) - h_{N,\varepsilon}(r,x,u_2,v_2)| \leq L[|u_1-u_2| + |v_1-v_2|],$$

Lemma A.1 implies that

$$\lim_{n\to\infty} \int_r^R \|V_s^{N,1/n} - V_s^N\|_2^2 ds = 0$$

for all $N = 1, 2, ...$ Hence there are sub-sequences $(n_l^N)_{l=1}^\infty$ such that

$$\lim_{l\to\infty} |V_s^{N,1/n_l^N} - V_s^N| = 0 \quad \mathbb{Q}\times\lambda \quad a.e.$$

and a Borel set $\mathcal{N}_N \subseteq [0,T]$ of Lebesgue measure zero such that

$$V_s^{N,1/n_l^N} \to_l V_s^N \quad \text{a.s. for} \quad s \notin \mathcal{N}_N.$$

Applying Fatou's lemma on the left-hand side of (27) and dominated convergence on the right-hand side (note that $|v_\varepsilon^x * \alpha^N| \leq N$ and that α is supposed to be continuous in x), we derive

$$\|V_t^N\|_p \leq d_2 \int_t^R \frac{\|\alpha^N(s,X_s)\|_p}{\sqrt{s-t}} ds \leq d_2 \int_t^R \frac{\|\alpha(s,X_s)\|_p}{\sqrt{s-t}} ds$$

for all $t \in [r,R] \backslash (\bigcup_{N'=1}^\infty \mathcal{N}_{N'} \cup \overline{\mathcal{N}})$. In the same way, Lemma A.1,

$$\int_r^R \|h^N(s,X_s,U_s,V_s) - h(s,X_s,U_s,V_s)\|_2 ds \to_N 0$$

and $|h^N(s,x,u_1,v_1) - h^N(s,x,u_2,v_2)| \leq L[|u_1-u_2| + |v_1-v_2|]$ give

$$\lim_{N\to\infty} \int_r^R \|V_s^N - V_s\|_2^2 ds = 0$$

and the existence of a subsequence $(N_k)_{k=1}^\infty$ such that

$$\lim_{k\to\infty} |V_s^{N_k} - V_s| = 0 \quad \mathbb{Q}\times\lambda \quad a.e.$$

Hence there is some $\mathcal{N}_0 \subseteq [r,R]$ of Lebesgue measure zero such that

$$V_s^{N_k} \to_k V_s \quad \text{a.s. for} \quad s \notin \mathcal{N}_0.$$

Again applying Fatou's lemma gives that

$$\|V_t\|_p \leq d_2 \int_t^R \frac{\|\alpha(s,X_s)\|_p}{\sqrt{s-t}} ds$$

for all $N = 1, 2, ...$ and $t \in [r,R] \backslash (\bigcup_{N'=0}^\infty \mathcal{N}_{N'} \cup \overline{\mathcal{N}})$. □

B Appendix

Proposition B.1 ([9, pp. 260, 72, 74, 44])**.** *For b, σ satisfying ($A_{b,\sigma}$), there exists a continuous transition density*

$$\Gamma : \{(t, x, s, \xi) : 0 \le t < s \le T \text{ and } x, \xi \in \mathbb{R}^d\} \to (0, \infty)$$

such that $\mathbb{P}(X_s^{t,x} \in B) = \int_B \Gamma(t, x; s, \xi) d\xi$ *for* $0 \le t < s \le T$ *and* $B \in \mathcal{B}(\mathbb{R}^d)$*, where*

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r,$$

such that the following is satisfied:

(i) *For all multi-indices m and k with $|m|+2k \le 3$ one has that the derivatives $D_t^k D_x^m \Gamma(t, x; s, \xi)$ exists and are continuous on $[0, s) \times \mathbb{R}^d$ and that the differentiation can be done in any order.*

(ii) *For $0 \le t < s \le T$ and $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ one has*

$$\frac{\partial}{\partial t}\Gamma + \frac{1}{2}\langle A, D^2\Gamma\rangle + \langle b, \nabla_x \Gamma\rangle = 0$$

where $A = \sigma\sigma^*$ and $D^2 = \left(\frac{\partial^2}{\partial x_i \partial x_j}\right)_{i,j=1}^d$.

(iii) *For all multi-indices m with $|m| \le 3$ there exists a constant $c = c_m > 0$ such that for $0 \le t < s \le T$ and $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ one has that*

$$|D_x^m \Gamma(t, x; s, \xi)| \le c \, (s - t)^{-\frac{|m|}{2}} \gamma_{s-t}^d \left(\frac{x - \xi}{c}\right)$$

where $\gamma_t^d(\eta) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|\eta|^2}{2t}}$.

Remark B.2. The weights $N_R^{r,i,(t,x)}$ are essential so that we briefly recall their construction. For notational simplicity we let $t = 0$ and omit the superscripts (t, x). For $i = 1$ one has $N_R^{r,1} := \frac{1}{R-r}\left(\int_r^R (\sigma(s, X_s)^{-1}\nabla X_s \nabla X_r^{-1})^* dW_s\right)^*$ where $\nabla X_t = \nabla_x b(t, X_t)\nabla X_t dt + \nabla_x \sigma(t, X_t)\nabla X_t dW_t$ with $\nabla X_0 = I_{\mathbb{R}^d}$, the identity matrix (see, for example, [20, 17]). To consider $i = 2$ we follow [17] and let $0 \le r < R \le T$, $\rho := (r + R)/2$, $g : \mathbb{R}^d \to \mathbb{R}$ be a Borel measurable polynomially bounded function and F like in (1). For $k = 1, ..., d$ we have that $(\partial F/\partial x_k)(r, X_r) = \mathbb{E}(F(\rho, X_\rho)N_\rho^{r,1}(k)|\mathcal{F}_r)$ a.s. Applying the ∇-operator, which can be justified by standard methods, we derive that, a.s.

$$\begin{aligned}
& \nabla_x(\partial F/\partial x_k)(r, X_r)\nabla X_r \\
= \; & \mathbb{E}(\nabla_x F(\rho, X_\rho)\nabla X_\rho N_\rho^{r,1}(k) + F(\rho, X_\rho)\nabla N_\rho^{r,1}(k)|\mathcal{F}_r) \\
= \; & \mathbb{E}(\mathbb{E}(g(X_R)N_R^{\rho,1}|\mathcal{F}_\rho)\nabla X_\rho N_\rho^{r,1}(k) + \mathbb{E}(g(X_R)|\mathcal{F}_\rho)\nabla N_\rho^{r,1}(k)|\mathcal{F}_r).
\end{aligned}$$

Therefore we can take $N_R^{r,2}(k) := [N_R^{\rho,1}\nabla X_\rho N_\rho^{r,1}(k) + \nabla N_\rho^{r,1}(k)](\nabla X_r)^{-1}$ to obtain $\nabla_x(\partial F/\partial x_k)(r, X_r) = \mathbb{E}(g(X_R)N_R^{r,2}(k)|\mathcal{F}_r)$ a.s.

Proof of Proposition 2.12. Assume that we have the diffusions $X^1 = (X_t^1)_{t\in[0,T_1]}$ and $X^2 = (X_t^2)_{t\in[0,T_2]}$ starting in $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}^d$ respectively, satisfying our assumptions with the corresponding transition densities Γ_1 and Γ_2, and assume that they satisfy

$$\Gamma_1(t,x;s,\xi) \le M\Gamma_2(\mu t,\nu x;\mu s,\nu\xi)$$

for some $M,\mu,\nu > 0$ and all $x,\xi \in \mathbb{R}^d$ and $0 \le t \le s \le T_1$ and with $T_2 = \mu T_1$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a polynomially bounded Borel function. Then, for $x_2 = \nu x_1$,

$$\mathbb{E}|g(X_{T_1}^1) - \mathbb{E}(g(X_{T_1}^1)|\mathcal{F}_t)|^p \le 2^p \frac{M^3}{\nu^3}\mathbb{E}|\widetilde{g}(X_{\mu T_1}^2) - \mathbb{E}(\widetilde{g}(X_{\mu T_1}^2)|\mathcal{F}_{\mu t})|^p$$

with

$$\widetilde{g}(x) := g\left(\frac{x}{\nu}\right).$$

In fact, we have that

$$\begin{aligned}
& \mathbb{E}|g(X_{T_1}^1) - \mathbb{E}(g(X_{T_1}^1)|\mathcal{F}_t)|^p \\
\le\; & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g(\xi) - g(\eta)|^p \\
& \qquad \Gamma_1(0,x_1;t,x)\Gamma_1(t,x;T_1,\xi)\Gamma_1(t,x;T_1,\eta)dxd\xi d\eta \\
\le\; & M^3\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g(\xi) - g(\eta)|^p \\
& \qquad \Gamma_2(0,\nu x_1;\mu t,\nu x)\Gamma_2(\mu t,\nu x;\mu T_1,\nu\xi)\Gamma_2(\mu t,\nu x;\mu T_1,\nu\eta)dxd\xi d\eta \\
=\; & \frac{M^3}{\nu^3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\widetilde{g}(\xi) - \widetilde{g}(\eta)|^p \\
& \qquad \Gamma_2(0,x_2;\mu t,x)\Gamma_2(\mu t,x;T_2,\xi)\Gamma_2(\mu t,x;T_2,\eta)dxd\xi d\eta \\
\le\; & 2^p\frac{M^3}{\nu^3}\mathbb{E}|\widetilde{g}(X_{T_2}^2) - \mathbb{E}(\widetilde{g}(X_{T_2}^2)|\mathcal{F}_{\mu t})|^p
\end{aligned}$$

where we used relation (6). This implies our assertion by taking $(\Gamma_1, x_1, T_1) = (\Gamma, x_0, r_l)$ and $T_2 = T_1$, $\nu = 1/c_{(B.1)}$ and $X_t^2 = \nu x_0 + W_t$. □

References

[1] R. Avikainen, On irregular functionals of SDEs and the Euler scheme, Finance and Stochastics 2009(13)381-401.

[2] C. Bennet and R. Sharpley, *Interpolation of operators.* Academic Press, 1988.

[3] J. Bergh and J. Lőfstrőm, *Interpolation spaces. An introduction.* Springer, 1976.

[4] B. Bouchard, R. Elie and N. Touzi, Discrete-time approximation of BSDEs and probabilistic schemes for fully nonlinear PDEs, Radon Series Comp. Appl. Math 2009(8)1-34.

[5] B. Bouchard and N. Touzi, Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations, Stoch. Proc. Appl. 2004(111)175-206.

[6] P. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica, L_p solutions of backward stochastic differential equations, Stoch. Proc. Appl., 2003(108)109-129.

[7] J. Creutzig, T. Müller-Gronbach and K. Ritter, Free-knot spline approximation of stochastic processes, J. Complexity (2007)23867-889.

[8] F. Delarue, On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case, Stoch. Proc. Appl., 2002(99)209-286.

[9] A. Friedman, *Partial Differential Equations of Parabolic Type.* Prentice-Hall, 1964.

[10] C. Geiss and S. Geiss, On approximation of a class of stochastic integrals and interpolation, Stochastics and Stochastics Reports, 2004(76)339-362.

[11] S. Geiss and M. Hujo, Interpolation and approximation in $L_2(\gamma)$, Journal of Approximation Theory 144; 213-232 (2007).

[12] S. Geiss and A. Toivola, Weak convergence of error processes in discretizations of stochastic integrals and Besov spaces, Bernoulli, 2009(15)925-954.

[13] S. Geiss and A. Toivola, BMO and L_p-approximation of stochastic integrals. In preparation.

[14] E. Gobet and A. Makhlouf, L_2-time regularity of BSDEs with irregular terminal functions, Stoch. Proc. Appl., 2010(120)1105-1132.

[15] E. Gobet, J.-P. Lemor and X. Warin, A regression-based Monte Carlo method to solve backward stochastic differential equations, Ann. Appl. Probab. 2005(15)2172-2202.

[16] E. Gobet, J.-P. Lemor and X. Warin, Rate of convergence of an empirical regression method for solving generalized backward stochastic differential equations, Bernoulli 2006(12)889-916.

[17] E. Gobet and R. Munos, Sensitivity analysis using Itô-Malliavin calculus and martingales, and application to stochastic optimal control, SIAM J. Control Optim. 2005(43)1676-1713.

[18] F. Hirsch, Lipschitz functions and fractional Sobolev spaces, Potential Analysis, 1999, (11), 415-429.

[19] Y. Hu, J. Ma, Nonlinear Feynman-Kac formula and discrete-functional-type BSDEs with continuous coefficients, Stoch. Proc. Appl., 2004(112)23-51.

[20] J. Ma, J. Zhang, Representation theorems for backward stochastic differential equations, Ann. Appl. Prob., 2002(12)1390-1418.

[21] A. Toivola, On fractional smoothness and approximations of stochastic integrals, PhD-Thesis, Department of Mathematics and Statistics, University of Jyväskylä, 2009.

[22] A. Toivola, Interpolation and approximation in L_p, Preprint 380, Department of Mathematics and Statistics, University of Jyväskylä, 2009.

[23] J. Zhang, A numerical scheme for BSDEs, Ann. Appl. Prob. 2004(14)459-488.

[24] J. Zhang, Representation of solutions to BSDE's associated with a degenerate FSDE, Ann. Appl. Prob., 2005(15)1798-1831.

[25] Y. Zhang and W. Zheng, Discretizing a backward stochastic differential equation, Internat. J. Math. Math. Sci. 2002(32)103-116.